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Qualitative Behavior Of Second Order Difference Equation with Non Positive Neutral Term

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Abstract:

The oscillation of second order difference equations with a nonlinear nonpositive neutral component is the subject of this study. We come up with a sufficient condition that guarantees that all solutions to the examined equation are either oscillatory or going towards zero. Through examples, the improvement of our primary findings is demonstrated.

Keywords: Oscillatory, neutral term, non positive, second order.

MSC: 39A10.

1. Introduction

In this article, we study some oscillatory manners of a second order non linear non positive neutral delay difference equation of the form

$$\Delta(r(\wp)\Delta(z(\wp) - p(\wp)z^{\gamma_1}(\tau_1(\wp)))) + q(\wp)z^{\gamma_2}(\sigma_1(\wp)), \quad \wp \geq \wp_0 > 0. \qquad (1)$$

subject to the restrictions outlined below :

(R1) γ_2 and $0 < \gamma_1 \leq 1$ are ratio of odd positive integers;

(R2) $\{r(\wp)\}, \{q(\wp)\}$ and $\{p(\wp)\}$ are positive real sequences such that $0 < p(\wp) \leq p < 1, \forall \wp \geq \wp_0$ and

(R3) σ_1 and τ_1 are positive integers with $\tau_1\wp) \leq \wp, \Delta\tau_1(\wp) > 0$, $\sigma_1(\wp) \leq \wp, \Delta\sigma_1(\wp) > 0$, $\lim_{\wp\to\infty}\tau_1(\wp) = \lim_{\wp\to\infty}\sigma_1(\wp) = \infty$.

A real sequence $\{z(\wp)\}$ is said to be a solution of (1) if it is defined for all $\wp \geq \wp$_0. A nontrivial solution of (1) is called oscillatory if it is neither eventually positive nor eventually negative. Otherwise, the solution is said to be non oscillatory. An equation is oscilltory if all its solutions oscillate.

Since neutral type equations are prevalent in the study of economics, mathematical biology, and many other fields of mathematics, determining oscillation conditions for these equations has garnered a lot of attention in recent years. (see for example $[1] - [9]$) . To the best of our knowledge, there are no results in the literature that guarantee that all solutions for the second order difference equation are just oscillatory. This conclusion is drawn from a review of the literature. All results ($[10] - [14]$) established for neutral type difference equations are guaranteed that every solution is either oscillatory or tends to zero monotonically. In order to define conditions for the oscillation of all solutions under the following condition, the authors considered (1) with $p(\wp) < 0$,

$$\sum_{i=\wp_0}^{\infty} \frac{1}{r(i)} = \infty \tag{2}$$

In this article we arrive at some new oscillation results.

2. Oscillatory Results

We begin with the following Lemmas, which are critical in establishing our key results.

We represent

$$s(\wp) = z(\wp) - p(\wp)z^{\gamma_1}(\tau_1(\wp)),$$
$$Y(\wp) = \sum_{i=\wp_1}^{\wp-1} \frac{1}{r(i)},$$

for every $\wp \geq \wp_1 \geq \wp_0$.

Lemma 2.1. Let (2) hold and if z is a positive solution of (1), then the corresponding function s meets one of the following two requirements :

(I) $s(\wp) > 0, \Delta s(\wp) > 0$ and $\Delta(r(\wp)\Delta s(\wp)) < 0$;

(II) $s(\wp) < 0, \Delta s(\wp) > 0$ and $\Delta(r(\wp)\Delta s(\wp)) < 0$,

for all $\wp \geq \wp_1$, where $\wp_1 \geq \wp_0$ is sufficiently large.

Proof. It is sufficient to state and prove the results for positive solutions. Because the proof of the other case is same.

Suppose that $z(\wp) > 0, z(\tau_1(\wp)) > 0$ and $z(\sigma_1(\wp)) > 0$ for every $\wp \geq \wp_1$ for some $\wp_1 \geq \wp_0$.

By the representation of $s(\wp)$ and (1), we get

$$\Delta(r(\wp)\Delta s(\wp)) = -q(\wp)z^{\gamma_2}(\sigma_1(\wp)) < 0. \tag{3}$$

Hence $r(\wp)(\Delta s(\wp))$ is decreasing and of one sign for large $\wp$, that means, $\exists\ \wp_2 \geq \wp_1$ and $\Delta s(\wp) > 0$

(or) $\Delta s(\wp) < 0$ for all $\wp \geq \wp_2$.

If $\Delta s(\wp) < 0$ for $\wp \geq \wp_2$, then $r(\wp)(\Delta s(\wp) \leq -d_1$ for $\wp \geq \wp_2$ where $d_1 = -r(\wp_2)\Delta s(\wp_2) > 0$. Then, we obtain

$$s(\wp) \leq s(\wp_2) - d_1 \sum_{i=\wp_2}^{\wp-1} \frac{1}{r(i)}.$$

By the condition (2), the above inequality implies $\lim_{\wp\to\infty} s(\wp) = -\infty$. We will now examine each of the next two situations separately.

Case (I): If z is unbounded, then $\exists$ a *sequence* $\{\wp_n\}$ such that $\lim_{n\to\infty} \wp_n = \infty$ and $\lim_{n\to\infty} z(\wp_n) = \infty$, where $z(\wp_n) = \max\{z(i), \wp_0 \leq i \leq \wp_n\}$.

Since $\lim_{\wp\to\infty} \tau_1(\wp) = \infty, \tau_1(\wp_n) > \wp_0$ for large $\wp$ and $\tau_1(\wp) \leq \wp$, then we obtain

$$z(\tau_1(\wp_n)) = \max\{z(i): \wp_0 \le i \le \tau_1(\wp_n)\}$$
$$\le \max\{z(i): \wp_0 \le i \le \wp_n\} = z(\wp_n).$$

That is, $z(\tau_1(\wp_n)) \le z(\wp_n)$.

Consequently,

$$s(\wp_n) = z(\wp_n) - p(\wp_n) z^{\gamma_1}(\tau_1(\wp_n))$$
$$\ge z(\wp_n)[1 - p(\wp_n) z^{\gamma_1 - 1}(\wp_n)] \to \infty$$

as $n \to \infty$, since $\gamma_1 \in (0,1]$ and $p(\wp)$ is bounded, which contradicts $\lim_{\wp\to\infty} s(\wp) = -\infty$.

Case (II): If z is bounded, then s is also bounded, because $p(\wp)$ is bounded, which contradicts that $\lim_{\wp\to\infty} s(\wp) = -\infty$. So $s(\wp)$ fulfills one of the cases (I) and (II).

Lemma 2.2. Let the condition (2) be true. Assume z be a positive solution of (1) there exists case (I) of

Lemma 2.1. Then

$$z(\wp) > s(\wp) > Y(\wp) r(\wp) \Delta s(\wp) \tag{4}$$

for $\wp \ge \wp_1$ and $s(\wp)/Y(\wp)$ is eventually decreasing.

Proof. By the representation of $s(\wp)$ and above (R_2), we can write $z(\wp) > s(\wp)$ for $\wp \ge \wp_1 \ge \wp_0$.

From the case (I), we get

$$s(\wp) = s(\wp_1) + \sum_{i=\wp_1}^{\wp-1} \frac{r(i)\Delta s(i)}{r(i)},$$
$$> Y(\wp) r(\wp) \Delta s(\wp),\ \wp \ge \wp_1. \tag{5}$$

Also,

$$\Delta\left(\frac{s(\wp)}{Y(\wp)}\right) = \frac{Y(\wp) r(\wp) \Delta s(\wp) - s(\wp)}{r(\wp) Y(\wp) Y(l+1)}$$
$$< 0,\ \wp \ge \wp 1.$$

Thus $\left\{\frac{s(\wp)}{Y(\wp)}\right\}$ is strictly decreasing for all $\wp \ge \wp_1$.

Theorem 2.3. Let $\gamma_2 < \gamma_1, \sigma_1(\wp) < \tau_1(\wp)$ and condition (2) hold. If

$$\sum_{\wp 1}^{\infty} q(\wp) Y^{\gamma_2}(\sigma_1(\wp)) = \infty \tag{6}$$

and

$$\lim_{\wp\to\infty} \sup \sum_{j=\tau_1^{-1}(\sigma_1(\wp))}^{\wp-1} \frac{1}{r(j)} \sum_{u=j_3}^{j-1} \frac{q(u)}{p^{\frac{\gamma_2}{\gamma_1}}\left(\tau_1^{-1}(\sigma_1(u))\right)} > 0\ , \tag{7}$$

then every solution of equation (1) is oscillatory.

Proof. Let z be a non oscillatory solution of (1). Then $z(\wp) > 0, z(\sigma_1(\wp)) > 0, z(\tau_1(\wp)) > 0, \wp \ge \wp_1 \ge \wp_0$. By Lemma 2.1, the corresponding function $s(\wp)$ fullfills either case (I) or case (II).

First, we assume that $s(\wp)$ satisfie s case (I). From the representation of $s(\wp)$, we get

$$\begin{aligned} z(\wp) &\geq s(\wp), \\ z^{\gamma_2}(\sigma_l(\wp)) &\geq s^{\gamma_2}(\sigma_l(\wp)). \end{aligned}$$

Applying above inequality in (1), we get

$$\Delta(r(\wp)\Delta s(\wp)) + q(\wp)s^{\gamma_2}(\sigma_1(\wp)) \leq 0. \tag{8}$$

Substituting (4) in (8) and taking $y(\wp) = r(\wp)\Delta s(\wp)$, we clear that $y(\wp)$ is a positive solution of the inequality

$$\Delta y(\wp) + q(\wp)Y^{\gamma_2}(\sigma_1(\wp))(r(\wp)\Delta s(\wp))^{\gamma_2} \leq 0,$$

$$\Delta y(\wp) + q(\wp)Y^{\gamma_2}(\sigma_1(\wp))y^{\gamma_2}(\sigma_1(\wp)) \leq 0,\ \wp \geq \wp_1 \tag{9}$$

On the other hand, from [6], we can see that condition (6) assures that (9) has no eventually positive solution, which is contradiction .

Next , assume that $s(\wp)$ satisfies case (II) of Lemma 2.1.

Then, by the representation of $s(\wp)$, we get

$$z(\tau_1(\wp)) > \left(\frac{-s(\wp)}{p(\wp)}\right)^{\frac{1}{\gamma_1}} . \tag{10}$$

Applying (10) in (1), we get

$$\Delta(r(\wp)\Delta s(\wp)) - \frac{1}{p^{\frac{\gamma_2}{\gamma_1}}\left(\tau_1^{-1}(\sigma_1(\wp))\right)} q(\wp)s^{\frac{\gamma_2}{\gamma_1}}\left(\tau_1^{-1}(\sigma_1(\wp))\right) \leq 0. \tag{11}$$

Since $s(\wp)$ is negative and increasing, we obtain $\lim_{\wp\to\infty} s(\wp) = c_1 \leq 0$. We prove that $c_1 = 0$. If not, then $c_1 < 0$ and $s(\wp) \leq c_1$ and $s\left(\tau_1^{-1}(\sigma_1(\wp))\right) \leq c_1$ for large $\wp$. Therefore,

$$s^{\frac{\gamma_2}{\gamma_1}}\left(\tau_1^{-1}(\sigma_1(\wp))\right) \leq c_1^{\frac{\gamma_2}{\gamma_1}} \tag{12}$$

Summing (11) from $\wp$ to ∞ and using (12), we get

$$r(\wp)\Delta s(\wp) - r(\wp_1)\Delta s(\wp_1) \leq \sum_{i=l}^{\infty} \frac{q(i)}{p^{\frac{\gamma_2}{\gamma_1}}\left(\tau_1^{-1}(\sigma_1(i))\right)} s^{\frac{\gamma_2}{\gamma_1}}\left(\tau_1^{-1}(\sigma_1(i))\right),$$

$$-r(\wp)\Delta s(\wp) \leq c_1^{\frac{\gamma_2}{\gamma_1}} \sum_{i=l}^{\infty} \frac{q(i)}{p^{\frac{\gamma_2}{\gamma_1}}\left(\tau_1^{-1}(\sigma_1(i))\right)}.$$

Again summing from $\wp_1$ to ∞, we have

$$s(\wp_1) \leq c_1^{\frac{\gamma_2}{\gamma_1}} \sum_{j=\wp_1}^{\infty} \frac{1}{r(j)} \sum_{m=j_1}^{\infty} \frac{q(m)}{p^{\frac{\gamma_2}{\gamma_1}}\left(\tau_1^{-1}(\sigma_1(m))\right)}$$

which is contradiction with (7) and from (7), we claim

$$\lim_{\wp\to\infty}\sup\sum_{j=\wp_1}^{\infty}\frac{1}{r(j)}\sum_{m=j_1}^{\infty}\frac{q(m)}{p^{\frac{\gamma_2}{\gamma_1}}\left(\tau_1^{-1}(\sigma_1(m))\right)}=\infty.$$

Thus, $\lim_{\wp\to\infty}s(\wp)=0$ and $s(\wp)$ is negative and increasing.

Summing (11) from $\wp_2$ to $\wp-1$ for $\wp>i$, we get

$$-r(\wp_2)(\Delta s(\wp_2))\leq\sum_{s=\wp_2}^{\wp-1}\frac{q(i)}{p^{\frac{\gamma_2}{\gamma_1}}\left(\tau_1^{-1}(\sigma_1(i))\right)}s^{\frac{\gamma_2}{\gamma_1}}\left(\tau_1^{-1}(\sigma_1(i))\right)$$

Again summing from $\tau_1^{-1}(\sigma_1(\wp))$ to $\wp-1$ and using $s(\wp)$ is increasing and we have

$$s\left(\tau_1^{-1}(\sigma_1(\wp))\right)-s(\wp)\leq s^{\frac{\gamma_2}{\gamma_1}}\left(\tau_1^{-1}(\sigma_1(\wp))\right)\sum_{j=\tau_1^{-1}(\sigma_1(\wp))}^{\wp-1}\frac{1}{r(j)}\sum_{m=j_3}^{j-1}\frac{q(m)}{p^{\frac{\gamma_2}{\gamma_1}}\left(\tau_1^{-1}(\sigma_1(m))\right)}$$

or

$$\frac{s\left(\tau_1^{-1}(\sigma_1(\wp))\right)}{s^{\frac{\gamma_2}{\gamma_1}}\left(\tau_1^{-1}(\sigma_1(\wp))\right)}\geq\sum_{j=\tau_1^{-1}(\sigma_1(\wp))}^{\wp-1}\frac{1}{r(j)}\sum_{m=j_3}^{j-1}\frac{q(m)}{p^{\frac{\gamma_2}{\gamma_1}}\left(\tau_1^{-1}(\sigma_1(m))\right)}. \tag{13}$$

Since

$$\frac{s\left(\tau_1^{-1}(\sigma_1(\wp))\right)}{s^{\frac{\gamma_2}{\gamma_1}}\left(\tau_1^{-1}(\sigma_1(\wp))\right)}=\left|s\left(\tau_1^{-1}(\sigma_1(\wp))\right)\right|^{1-\frac{\gamma_2}{\gamma_1}}$$

and $1-\frac{\gamma_2}{\gamma_1}>0$, we get

$$\lim_{\wp\to\infty}\sup\sum_{j=\tau_1^{-1}(\sigma_1(\wp))}^{\wp-1}\frac{1}{r(j)}\sum_{m=j_3}^{j-1}\frac{q(m)}{p^{\frac{\gamma_2}{\gamma_1}}\left(\tau_1^{-1}(\sigma_1(m))\right)}\leq 0$$

which contradicts (7).

Theorem 2.4. Assume $\gamma_2=1$ and condition (2) holds. If

$$\lim_{t\to\infty}\inf\sum_{\sigma_1(\wp)}^{\wp-1}q(s)Y(\sigma_1(s))>\frac{1}{e}\ , \tag{14}$$

then every solution of (1) is either oscillatory or tends to zero as $\wp\to\infty$.

Proof. We assume that a non-oscillatory solution z of (1), $z(\wp)>0,\ z\,(\sigma_1(\wp))>0,\ z(\tau_1(\wp))>0,\ \wp\geq\wp_1\geq\wp_0$ and that for s one of the case (I) and case (II) holds.

Assume that $s(\wp)$ meets case (I) of Lemma 2.1 and from the proof of case (I) of theorem 2.1, we have for $\gamma_2=1$ that $y(\wp)=r(\wp)\Delta s(\wp)$ is a positive solution of the inequality

$$\Delta y(\wp)+q(\wp)Y(\sigma_1(\wp))y(\sigma_1(\wp))\leq 0. \tag{15}$$

On the other hand, from [6], we can notice that equation (14) guarantees that (15) has no positive solution, which implies contradiction.

Let $s(\wp)$ meets case (II) of Lemma 2.1. From this $s(\wp) < 0$ and $\Delta s(\wp) > 0$ and also $\lim_{\wp\to\infty} s(\wp) = c_1 \leq 0$, where c_1 is a constant. i.e. s is bounded and as in the proof of Lemma 2.1, we can say that z is also bounded.

Therefore, $\lim_{\wp\to\infty} z(\wp) = m_1, 0 \leq m_1 < \infty$. We claim that $m_1 = 0$. Suppose $m_1 > 0$, there is a sequence $\{\wp_n\}$ such that $\lim_{n\to\infty} \wp_n = \infty$ and $\lim_{n\to\infty} z(\wp_n) = m_1$.

Thus

$$s(\wp_n) = z(\wp_n) - p(\wp_n) z^{\gamma_1}\big(\tau_1(\wp_n)\big),$$

$$z\big(\tau_1(\wp_n)\big) = \frac{\big(z(\wp_n) - s(\wp_n)\big)^{\frac{1}{\gamma_1}}}{p^{\frac{1}{\gamma_1}}(\wp_n)}.$$

Taking $n \to \infty$, we get

$$\begin{aligned} m_1 &\geq \lim_{n\to\infty} z\big(\tau_1(\wp_n)\big) \\ &\geq \left(\frac{m_1}{p}\right)^{\frac{1}{\gamma_1}} \end{aligned}$$

We conclude that $m_1 = 0$, because of $p \in (0,1)$, that is $\lim_{n\to\infty} z(\wp) = 0$.

3. Examples

Example 3.1. Examine second order neutral delay difference equation

$$\Delta\left(\wp\Delta\left(z(\wp) - pz^{\frac{1}{3}}\left(\frac{\wp}{2}\right)\right)\right) + 8\wp\left(\frac{\wp}{3}\right) = 0, \quad \wp \geq 1, \tag{16}$$

where $p \in (0,1)$ which is a constant. Here $r(\wp) = \wp, p(\wp) = p, q(\wp) = 8\wp, \tau_1(\wp) = \frac{\wp}{2}, \sigma_1(\wp) = \frac{\wp}{3}$ for $\wp \geq \wp_1 = 1, \gamma_1 = 1/3, \gamma_2 = 1/5$ and $Y(\wp) = \frac{1-\wp}{\wp}$. Clearly, these calculations shows that above conditions (6) and (7) are fullfilled. So that by Theorem 2.3, every solution of (16) is oscillatory.

Example 3.2. Examine second order neutral delay difference equation

$$\Delta\left(\frac{1}{\wp}\Delta\left(z(\wp) - pz^{\frac{1}{3}}\left(\frac{\wp}{2}\right)\right)\right) + \wp z\left(\frac{\wp}{3}\right) = 0, \quad \wp \geq 1, \tag{17}$$

where $p \in (0,1)$ which is a constant. Here $r(\wp) = \frac{1}{\wp}, p(\wp) = p, q(\wp) = \wp, \tau_1(\wp) = \frac{\wp}{2}, \sigma_1(\wp) = \frac{\wp}{3}$ for $\wp \geq \wp_1 = 1, \gamma_1 = 1/3, \gamma_2 = 1$ and $Y(\wp) = 1 - \wp$. Each and every conditons of Theorem 2.4 with $\gamma_2 = 1$ are satisfied, so the equation (17) is oscillatory.

4. Conclusion

The solutions of nonlinear equations behave in peculiar ways and these ways can be developed by means of different strategies included in the method. An attempt was made here to establish the sufficient conditions with the fact that the solution space of nonlinear non positive neutral term of difference equation is reducing to the solution of its limiting equation and we assumed with $\gamma_2 = 1$. By these discussions, (1) is oscillatory or asymptotically zero as $\wp \to \infty$.

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