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RESEARCH ARTICLE

Qualitative Behavior of Fourth Order Neutral Difference Equations

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ABSTRACT

"The goal of this paper is to illustrate the qualitative behaviour of a fourth order difference equation with neutral terms of the form

$\Delta\left(s_1(\vartheta)\left(\Delta^3 q(\vartheta)\right)^{\beta_1}\right) = s_2(\vartheta)y^{\beta_2}(\vartheta - n + 1) + s_3(\vartheta)y^{\beta_3}(\vartheta + n^*)$

where $q(\vartheta) = y(\vartheta) - s_4(\vartheta)y^{\beta_4}(\vartheta - k)$ Here $\beta_1, \beta_2, \beta_3, \beta_4$ are the ratios of odd positive integers $\beta_1 \geq 1$, s_1, s_2, s_3, s_4 are positive sequences and $n, n^*, k \in N$ are such that $n > 3, n^* > 3$, $k < n - 2$. With the help of comparison techniques, we are able to acquire some novel oscillations results. Examples are given to illustrate the importance of the discoveries.

Keywords : comparison techniques, fourth order, neutral terms, oscillation.

INTRODUCTION

Due to the fact that neutral difference equations are used in the study of economics, mathematical biology, and many other areas of mathematics, the issue of establishing oscillation phenomena for these equations has drawn a lot of attention in recent years [1],[2],[11],[12],[17]. The sources cited there as well as [4], [21], [22], [23] provide some fascinating new findings on the oscillatory behavior of second-order differential equations. A examination of the literature reveals that every conclusion made for fourth order difference equations with neutral terms ensures that each solution oscillates or monotonically approaches to zero. As far as we are aware, no conclusions have been drawn for fourth order neutral difference equations that suggest that all solutions are just oscillatory. This study's goal is to provide the equation some revised oscillation restrictions as a result.

$$\Delta\left(s_1(\vartheta)\left(\Delta^3 q(\vartheta)\right)^{\beta_1}\right) = s_2(\vartheta)y^{\beta_2}(\vartheta - n + 1) + s_3(\vartheta)y^{\beta_3}(\vartheta + n^*) \quad (1)$$

where $q(\vartheta) = y(\vartheta) - s_4(\vartheta)y^{\beta_4}(\vartheta - k)$ via comparing first order equations with known oscillatory phenomena, or by comparing second-order difference equations with neutral terms. The reader can refer to [5], [6], [7] for relevant results on oscillation theory of applications.

The following conditions are always considered to hold:

(i) $\beta_1, \beta_2, \beta_3, \beta_4$ are the ratios of odd positive integers with $\beta_1 \geq 1$;

(ii) s_1, s_2, s_3, s_4 are positive sequences;

(iii) $n, n^*, k \in N$ are such that $n > 3, n^* > 3, k < n - 2$.

A solution to (1) is said to be oscillatory if it is neither eventually negative nor finally positive. If not, it is regarded as non-oscillatory. Equation (1) is oscillatory if and only if all of its solutions are oscillatory." The aim of this work is to generate adequate conditions for (1) to oscillate whenever $\beta_4 < 1$ and subject the assumption

$S_1(\vartheta, \vartheta_1) \to \infty$ as $\vartheta \to \infty$ where $S_1(r, u) = \sum_{\varrho=u}^{r-1} \frac{1}{s_1^{\frac{1}{\beta_1}}(\varrho)}$ (2)

AUXILIARY RESULTS

Lemma 2.1 (see [7], Lemma 1 and 19 , Lemma 2.2).

(I) If the first order delay difference inequality

$\Delta q(\vartheta) - s_2(\vartheta) f(q(\vartheta - n + 1)) \leq 0$

has an eventually positive solution, then so does the corresponding delay difference equation.

(II)If the first order advanced difference inequality

$\Delta q(\vartheta) - s_2(\vartheta) f\big(q(\vartheta - n^*)\big) \geq 0$

has an eventually positive solution, then so does the corresponding advanced difference equation.

Lemma 2.2. (see [9])

If $X, Y \geq 0$, then

$X^\gamma + (\gamma - 1)Y^\gamma - \gamma X Y^{\gamma-1} \geq 0 \quad \text{for } \gamma > 1$ (3)

and

$X^\gamma + (1 - \gamma)Y^\gamma - \gamma X Y^{\gamma-1} \leq 0$ for $0 < \gamma < 1$. (4)

Lemma 2.3. Assume (2). Then $\Delta Q(\vartheta) > 0$ eventually, where

$$Q := s_1\left(\Delta^3 q\right)^{\beta_1} \tag{5}$$

suggests that one of the below four scenarios occurs:

Case (I). $q(\vartheta) > 0$, $\Delta q(\vartheta) > 0$, $\Delta^2 q(\vartheta) > 0$ $\Delta^3 q(\vartheta) > 0$;

Case (II). $q(\vartheta) > 0$, $\Delta q(\vartheta) > 0$, $\Delta^2 q(\vartheta) > 0$ $\Delta^3 q(\vartheta) < 0$;

Case (III). $q(\vartheta) < 0$, $\Delta q(\vartheta) < 0$, $\Delta^2 q(\vartheta) < 0$ $\Delta^3 q(\vartheta) < 0$;

Case (IV). $q(\vartheta) < 0$, $\Delta q(\vartheta) > 0$, $\Delta^2 q(\vartheta) < 0$ $\Delta^3 q(\vartheta) < 0$.

Proof. From (5), we can find $\vartheta_0 \in N_0$ such that

$$\Delta Q(\vartheta) > 0 \qquad \forall \vartheta \geq \vartheta_0. \tag{6}$$

We suppose that there exists $\vartheta_1 \geq \vartheta_0$ with

$$Q(\vartheta_1) > 0 \tag{7}$$

From (6) and (7), we get, $\forall \vartheta \geq \vartheta_1$.

$$Q(\vartheta) = Q(\vartheta_1) + \sum_{\varrho=\vartheta_1}^{\vartheta-1} \Delta Q(\varrho) \geq Q(\vartheta_1) > 0$$

Thus,

$$\Delta^3 q(\vartheta) > 0 \qquad \forall \vartheta \geq \vartheta_1. \tag{8}$$

From (6) and (7) we obtain, for $\vartheta \geq \vartheta_1$

$$\Delta^2 q(\vartheta) = \Delta^2 q(\vartheta_1) + \sum_{\varrho=\vartheta_1}^{\vartheta-1} \Delta^3 q(\varrho) = \Delta^2 q(\vartheta_1) + \sum_{\varrho=\vartheta_1}^{\vartheta-1} \frac{Q^{\frac{1}{\beta_1}}(\varrho)}{s_1^{\frac{1}{\beta_1}}(\varrho)}$$

$$\geq \Delta^2 q(\vartheta_1) + \sum_{\varrho=\vartheta_1}^{\vartheta-1} \frac{Q^{\frac{1}{\beta_1}}(\vartheta_1)}{s_1^{\frac{1}{\beta_1}}(\vartheta_1)} \geq \Delta^2 q(\vartheta_1) + Q^{\frac{1}{\beta_1}}(\vartheta_1) S_1(\vartheta, \vartheta_1) \to \infty \quad \text{as } \vartheta \to \infty.$$

because of (2).
Therefore, there exists $\vartheta_2 \geq \vartheta_1$ with
$\Delta^2 q(\vartheta) > 0 \quad \forall \vartheta \geq \vartheta_2$. (9)
From (8) and (9) we get, $\forall \vartheta \geq \vartheta_2$.

$$\Delta q(\vartheta) = \Delta q(\vartheta_2) + \sum_{\varrho=\vartheta_2}^{\vartheta-1} \Delta^2 q(\varrho) \geq \Delta q(\vartheta_2) + \sum_{\varrho=\vartheta_2}^{\vartheta-1} \Delta^2 q(\vartheta_2)$$

$$= \Delta q(\vartheta_2) + (\vartheta - \vartheta_2)\Delta^2 q(\vartheta_2) \to \infty \quad \text{as } \vartheta \to \infty.$$

Hence, there exists $\vartheta_1 > \vartheta_2$ with $\Delta q(\vartheta) > 0 \quad \forall \vartheta \geq \vartheta_3$. (10)
From (9) and (10) we obtain for $\vartheta \geq \vartheta_3$,

$$q(\vartheta) = q(\vartheta_3) + \sum_{\varrho=\vartheta_3}^{\vartheta-1} \Delta q(\varrho) \geq q(\vartheta_3) + \sum_{\varrho=\vartheta_3}^{\vartheta-1} \Delta q(\vartheta_3)$$

$$= q(\vartheta_3) + (\vartheta - \vartheta_3)\Delta q(\vartheta_3) \to \infty \quad \text{as } \vartheta \to \infty.$$

Hence, there exists $\vartheta_4 \geq \vartheta_3$ with
$q(\vartheta) > 0 \quad \forall \vartheta \geq \vartheta_4$ (11)
By (8)-(11), we get

$$q(\vartheta) > 0, \Delta q(\vartheta) > 0, \Delta^2 q(\vartheta) > 0, \Delta^3 q(\vartheta) > 0 \forall \vartheta \geq \vartheta_4$$

Thus, Case (I) holds if (7) does not hold, then the only other possibilities is $q(\vartheta) < 0$ for $\vartheta \geq \vartheta_0$ and thus,

$$\Delta^3 q(\vartheta) < 0 \text{ for all } \vartheta \geq \vartheta_0 \tag{12}$$

we suppose that, there exists $\vartheta_1 \geq \vartheta_0$ with
$\Delta^2 q(\vartheta_1) < 0$ (13)
From (12) and (13) we get, $\forall \vartheta \geq \vartheta_1$

$$\Delta^2 q(\vartheta) = \Delta^2 q(\vartheta_1) + \sum_{\varrho=\vartheta_1}^{\vartheta-1} \Delta^3 q(\varrho) \leq \Delta^2 q(\vartheta_1) < 0$$

Hence

$$\Delta^2 q(\vartheta) < 0 \quad \forall\ \vartheta \geq \vartheta_1 \tag{14}$$

Now, from (12) and (13) we obtain for $\vartheta \geq \vartheta_1$

$$\Delta q(\vartheta) = \Delta q(\vartheta_1) + \sum_{\varrho=\vartheta_1}^{\vartheta-1} \Delta^2 q(\varrho) \leq \Delta q(\vartheta_1) + \sum_{\varrho=\vartheta_1}^{\vartheta-1} \Delta^2 q(\vartheta_1)$$

$$\leq \Delta q(\vartheta_1) + (\vartheta - \vartheta_1)\Delta^2 q(\vartheta_1) \to -\infty \quad \text{as } \vartheta \to \infty,$$

Hence there exists $\vartheta_2 \geq \vartheta_1$ with
$\Delta q(\vartheta) < 0 \quad \forall\ \vartheta \geq \vartheta_2$ (15)
Now, from (14) and (15) we can get, for $\vartheta \geq \vartheta_2$,

$$q(\vartheta) = q(\vartheta_2) + \sum_{\varrho=\vartheta_2}^{\vartheta-1} \Delta q(\varrho) \leq q(\vartheta_2) + \sum_{\varrho=\vartheta_2}^{\vartheta-1} \Delta q(\vartheta_2)$$

$$\leq q(\vartheta_2) + (\vartheta - \vartheta_2)\Delta q(\vartheta_2) \to -\infty \quad \vartheta \to \infty,$$

Thus, there exist $\vartheta_3 \geq \vartheta_2$ with
$q(\vartheta) < 0 \quad \forall\ \vartheta \geq \vartheta_3$ (16)
By equation (12)-(16), we get $q(\vartheta) < 0, \Delta q(\vartheta) < 0, \Delta^2 q(\vartheta) < 0, \Delta^3 q(\vartheta) < 0$.
Thus, Case III holds. Further, if (13) does not hold, then the only other possibility is

$$\Delta^2 q(\vartheta_1) > 0 \quad \forall\ \vartheta \geq \vartheta_0 \tag{17}$$

Suppose that, there exists $\vartheta_1 \geq \vartheta_0$ with

$$\Delta q(\vartheta_1) > 0 \tag{18}$$

Then, from (17) and (18) we get $\forall \vartheta_1 \geq \vartheta_0$,

$$\Delta q(\vartheta) = \Delta q(\vartheta_1) + \sum_{\varrho=\vartheta_1}^{\vartheta-1} \Delta^2 q(\varrho) \geq \Delta q(\vartheta_1) > 0.$$

Therefore, there exists $\vartheta_2 \geq \vartheta_1$ with

$$\Delta q(\vartheta) > 0 \ \forall\, \vartheta_1 \geq \vartheta_0 \tag{19}$$

Now, from (18) and (19)$\forall \vartheta \geq \vartheta_2$,

$$q(\vartheta) = q(\vartheta_2) + \sum_{\varrho=\vartheta_2}^{\vartheta-1} \Delta q(\varrho) \geq q(\vartheta_2) + \sum_{\varrho=\vartheta_2}^{\vartheta-1} \Delta q(\vartheta_2)$$

$$\geq q(\vartheta_2) + (\vartheta - \vartheta_2)\Delta q(\vartheta_2) \to \infty \text{ as } \vartheta \to \infty$$

Hence, there exists $\vartheta_1 \geq \vartheta_2$ with

$$q(\vartheta) > 0 \ \forall\, \vartheta \geq \vartheta_3. \tag{20}$$

By Equations (12),(17),(19),(20) we get, $q(\vartheta) > 0, \Delta q(\vartheta) > 0, \Delta^2 q(\vartheta) > 0, \Delta^3 q(\vartheta) < 0$. Case (II) is so upheld. In the event ϱ (15) does not hold, the sole option is

$$\Delta q(\vartheta) > 0 \ \ \forall\, \vartheta \geq \vartheta_0. \tag{21}$$

By Equation (12), (14),(16),(21) we get, $q(\vartheta) < 0, \Delta q(\vartheta) > 0, \Delta^2 q(\vartheta) < 0, \Delta^3 q(\vartheta) < 0$, Thus, Case (IV) holds. The rest the article is based on the assumption that

$$k_0, k_1, k_2, k_3, k_4 \in \mathbb{N} \quad \text{satisfying } 3k_0 < n^*, k_1 < n + 2, \text{ and } k_2 < k_3 \leq n + 1 - k. \tag{22}$$

Note 2.4:

1. Consider the constraints $n > 3, n^* > 3$, and $k < n - 2$. For example, one may use $k_0 = k_1 = k_2 = 1, k_3 = 2$and $k_4 = 4$.
2. Consider that $\vartheta + n^* - 3k_0 > \vartheta$. Since it is always possible $n^* - 3k_0 > 0$. Therefore, equations involving $\vartheta + n^* - 3k_0$ are advanced type. Furthermore $\vartheta + n^* + k_1 - 2 < \vartheta, \vartheta - n + k - 2 < \vartheta_1$, $\vartheta - n + k - 2 + k_1 < \vartheta$, always since $n + 2 - k_1 > 0, n + 2 - k > 0, n + 2 - k - k_3 > 0$. Therefore, equation containing $\vartheta - n + k_1 - 2$, $\vartheta - n + k - 2, \vartheta - n + k - 2 + k_3$are of delay type.

MAIN RESULTS

We will start looking at the new result.

Theorem 3.1.*Let* $\beta_4 < 1$. Assume that(i)-(iii), (2) and (22) hold. Suppose that there is a sequence $s: Q \to (0, \infty)$. Such that $\lim_{\vartheta\to\infty}(g_1(\vartheta)) = 0$, where

$$g_1(\vartheta) := (1 - \beta_4)\beta_4^{\frac{\beta_4}{1-\beta_4}} s^{\frac{\beta_4}{\beta_4-1}}(\vartheta) s_4^{\frac{1}{1-\beta_4}}(\vartheta). \tag{23}$$

Let $\theta_0, \theta_1, \theta_2 \in (0,1)$. If the first-order advanced difference equation

$$\Delta q(\vartheta) = \theta_0 z^{\frac{\beta_3}{\beta_1}}(\vartheta + n^* - 3k_0) \sum_{\varrho=\vartheta-k_0}^{\vartheta-1} \left(\sum_{r=t-k_0}^{\varrho-1} \left(\frac{1}{s_1(\varrho)} \sum_{\gamma=r-k_0}^{r-1} s_3(\gamma) \right)^{\frac{1}{\beta_1}} \right) \tag{24}$$

and the first order delay difference equation

$$\Delta W(\vartheta) + (\theta_1\theta_2 k)^{\beta_2} s_2(\vartheta)(\vartheta - n + 1)^{\beta_2}(\vartheta - n)^{\beta_2} W^{\frac{\beta_2}{\beta_1}}(\vartheta - n + k_1 - 2)[S_1(\vartheta - n + k_1 - 1, \vartheta - n - 1)]^{\beta_2} = 0 \tag{25}$$

$$\Delta W(\vartheta) + \frac{s_2(\vartheta)}{(s_4)^{\frac{\beta_2}{\beta_4}}(\vartheta - n + k + 1)} W^{\frac{\beta_2}{\beta_1\beta_4}}(\vartheta - n + k - 2) \left(\sum_{\varrho=\vartheta_1}^{\vartheta-n+k} \left(\sum_{r=t_1}^{t-n+k} S_1(r_1, \vartheta_1) \right) \right) = 0 \tag{26}$$

and

$$\Delta W(\vartheta) + \frac{s_2(\vartheta)}{s_4^{\frac{\beta_2}{\beta_4}}(\vartheta-n+k+1)}(k_2 c\theta_0)^{\frac{\beta_2}{\beta_4}}\left[W^{\frac{1}{\beta_1}}(\vartheta-n+k+k_3-2)\, S_1(\vartheta-n+k+k_3-1,\ \vartheta-n+k+k_3-1)\right]^{\frac{\beta_2}{\beta_4}} = 0 \quad (27)$$

are oscillatory, then so is (1)

Proof. Suppose thaty is a non oscillatory solution of (1), say $y(\vartheta) > 0, y(\vartheta-k) > 0,\ y(\vartheta-n+k) > 0, y(\vartheta+n^*) > 0$ eventually

$$\Delta\left(s_1(\vartheta)(\Delta^3 q(\vartheta)^{\beta_1}\right) = s_2(\vartheta)y^{\beta_2}(\vartheta-n+1) + s_3(\vartheta)y^{\beta_3}(\vartheta+n^*) > 0 \quad (28)$$

Therefore (5) is fulfilled, and only four cases (I), (II), (III) and (IV) can be done according to Lemma (2.3)

Case (I) and (II):

Using $\gamma = \beta_4 \in (0,1), X = s_4^{\frac{1}{\beta_4}}(\vartheta)y(\vartheta-k), Y = \left(\frac{1}{\beta_4}s(\vartheta)s_4^{\frac{-1}{\beta_4}}(\vartheta)\right)^{\frac{1}{\beta_4-1}}$ in (4) we obtain $-\left(s(\vartheta)y(\vartheta-k) - s_4(\vartheta)y^{\beta_4}(\vartheta-k)\right) \le g_1(\vartheta)$

$$\begin{aligned} y(\vartheta) &= q(\vartheta) + s_4(\vartheta)y^{\beta_4}(\vartheta-k) + s(\vartheta)y(\vartheta-k) - s(\vartheta)y(\vartheta-k) \\ &= q(\vartheta) + s(\vartheta)y(\vartheta-k) - \left(s(\vartheta)y(\vartheta-k) - s_4(\vartheta)y^{\beta_4}(\vartheta-k)\right) \\ &\ge q(\vartheta) + s(\vartheta)y(\vartheta-k) + g_1(\vartheta) \\ &\ge q(\vartheta)\left[1 + \frac{s(\vartheta)y(\vartheta-k) + g_1(\vartheta)}{q(\vartheta)}\right] \end{aligned}$$

As q in both cases (I) and (II) is positive and non-decreasing, there exists $L > 0$, fulfilling of $q(\vartheta) > L$, and thus, we obtain

$$y(\vartheta) \ge \left(1 + \frac{s(\vartheta)y(\vartheta-k) + g_1(\vartheta)}{\vartheta}\right)q(\vartheta)$$

Then due to (23), there exists $k \in (0,1)$ such that

$$y(\vartheta) \ge kq(\vartheta) \quad (29)$$

eventually

So, we get

$$\Delta\left(s_1(\vartheta)\left(\Delta^3 q(\vartheta)\right)^{\beta_1}\right) \ge k^{\beta_2}s_2(\vartheta)q^{\beta_2}(\vartheta-n+1) + k^{\beta_3}s_3(\vartheta)q^{\beta_3}(\vartheta+n^*) \ge 0 \quad (30)$$

Case(I).Using (30), we get

$$\Delta\left(s_1(\vartheta)\left(\Delta^3 q(\vartheta)\right)^{\beta_1}\right) \ge k^{\beta_3}s_3(\vartheta)q^{\beta_3}(\vartheta+n^*) \quad (31)$$

Summing (31) from $\vartheta - k_0$to $\vartheta - 1$, we have

$$\begin{aligned} s_1(\vartheta)\left(\Delta^3 q(\vartheta)\right)^{\beta_1} &= s_1(\vartheta-k_0)\left(\Delta^3 q(\vartheta-k_0)\right)^{\beta_1} + \sum_{\varrho=\vartheta-k_0}^{\vartheta-1} \Delta\left(s_1(\varrho)\left(\Delta^3 q(\varrho)\right)\right)^{\beta_1} \\ &\ge k^{\beta_3}\sum_{\varrho=\vartheta-k_0}^{\vartheta-1} s_3(\varrho)q^{\beta_3}(\varrho+n^*) \\ &\ge k^{\beta_3}q^{\beta_3}(\vartheta+n^*-k_0)\sum_{\varrho=\vartheta-k_0}^{\vartheta-1} s_3(\varrho) \end{aligned}$$

So we get,

$$\Delta^3 q(\vartheta) \ge k^{\frac{\beta_3}{\beta_1}}q^{\frac{\beta_3}{\beta_1}}(\vartheta+n^*-k_0)\left(\frac{1}{s_1(\vartheta)}\sum_{\varrho=\vartheta-k_0}^{\vartheta-1} s_3(\varrho)\right)^{\frac{1}{\beta_1}} \quad (32)$$

Summing (32) again from $\vartheta - k_0$to$\vartheta - 1$, we get

$$\begin{aligned} \Delta^2 q(\vartheta) &= \Delta^2 q(\vartheta-k_0) + \sum_{\varrho=\vartheta-k_0}^{\vartheta-1} \Delta^3 z(\varrho) \\ &\ge \sum_{\varrho=\vartheta-k_0}^{\vartheta-1} k^{\frac{\beta_3}{\beta_1}}q^{\frac{\beta_3}{\beta_1}}(\varrho+n^*-k_0)\left(\frac{1}{s_1(\varrho)}\sum_{r=t-k_0}^{\varrho-} s_3(r)\right)^{\frac{1}{\beta_1}} \end{aligned}$$

$\geq k^{\frac{\beta_3}{\beta_1}} q^{\frac{\beta_3}{\beta_1}}(\vartheta + n^* - 2k_0) \sum_{\varrho=\vartheta-k_0}^{\vartheta-1} \left(\frac{1}{s_1(\varrho)} \sum_{r=t-k_0}^{\varrho-} s_3(r)\right)^{\frac{1}{\beta_1}}$ (33)

Summing (33) again from $\vartheta - k_0$to $\vartheta - 1$, we get

$$\Delta q(\vartheta) = \Delta q(\vartheta - k_0) + \sum_{\varrho=\vartheta-k_0}^{\vartheta-1} \Delta^2 q(\vartheta)$$

$$\geq k^{\frac{\beta_3}{\beta_1}} \sum_{r=\vartheta-k_0}^{\vartheta-1} q^{\frac{\beta_3}{\beta_1}}(\varrho + n^* - 2k_0) \sum_{r=t-k_0}^{\varrho-} \left(\frac{1}{s_1(r)} \sum_{u=r-k_0}^{\vartheta-1} s_3(u)\right)^{\frac{1}{\beta_1}}$$

$$\geq k^{\frac{\beta_3}{\beta_1}} q^{\frac{\beta_3}{\beta_1}}(\vartheta + n^* - 3k_0) \sum_{\varrho=\vartheta-k_0}^{\vartheta-1} \left(\sum_{r=t-k_0}^{\varrho-} \left(\frac{1}{s_1(r)} \sum_{u=r-k_0}^{r-1} s_3(u)\right)^{\frac{1}{\beta_1}} \right)$$

Hence, we conclude that q is a positive and increasing solution of

$$\Delta q(\vartheta) - k^{\frac{\beta_3}{\beta_1}} q^{\frac{\beta_3}{\beta_1}}(\vartheta + n^* - 3k_0) \sum_{\varrho=\vartheta-k_0}^{\vartheta-1} \left(\sum_{r=t-k_0}^{\varrho-} \left(\frac{1}{s_1(r)} \sum_{u=r-k_0}^{r-1} s_3(u)\right)^{\frac{1}{\beta_1}} \right) \geq 0$$

while applying Lemma (2.1)II, (24) also has an eventually positive solution, which is a contradiction.

Case (II). Let

$W = -s_1\left(\Delta^3 q\right)^{\beta_1} > 0$ eventually (34)

By (30) we get

$-\Delta W(\vartheta) \geq k^{\beta_2} s_2(\vartheta) q^{\beta_2}(\vartheta - n + 1)$ (35)

we know that, eventually,

$$q(\vartheta) = q(\vartheta_1) + \sum_{\varrho=\vartheta_1}^{\vartheta-1} \Delta q(\varrho) \geq \sum_{\varrho=\vartheta_1}^{\vartheta-1} \Delta q(\varrho)$$

$$= (\vartheta - \vartheta_1)\Delta q(\vartheta - 1) = l\Delta q(\vartheta - 1)\left(1 - \frac{\vartheta_1}{\vartheta}\right)$$

Since $\frac{\vartheta_1}{\vartheta} \to 0$as $\vartheta \to \infty$, there exists $\theta_1 \in (0,1)$ such that

$q(\vartheta) \geq \vartheta\theta_1 \Delta q(\vartheta - 1)$ eventually (36)

Next, we have

$$\Delta q(\vartheta) = q(\vartheta_2) + \sum_{\varrho=\vartheta_2}^{\vartheta-1} \Delta^2 q(\varrho) \geq \sum_{\varrho=\vartheta_2}^{\vartheta-1} \Delta^2 q(\vartheta - 1)$$

$$= (\vartheta - \vartheta_2)\Delta^2 q(\vartheta - 1) = \Delta^2 z(\vartheta - 1)\left(1 - \frac{\vartheta_2}{\vartheta}\right)$$

Since $\frac{\vartheta_2}{\vartheta} \to 0$ as $\vartheta \to \infty$ there exists $\theta_2 \in (0,1)$ such that

$\Delta q(\vartheta) \geq \vartheta\theta_2 \Delta^2 q(\vartheta - 1)$ eventually (37)

Now, we get $a = \vartheta - n - 1, b = a + k_1 > a$.

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rom (34) and (28) we geteventually

$$\begin{aligned}0 &\le \Delta^2 q(b) = \Delta^2 q(a) + \textstyle\sum_{\varrho=a}^{b-1} \Delta^3 q(\varrho)\\ &= \Delta^2 q(a) - \textstyle\sum_{\varrho=a}^{b-1} \frac{W^{\frac{1}{\beta_1}}(\varrho)}{s_1^{\frac{1}{\beta_1}}(\varrho)}\\ &\le \Delta^2 q(a) - \textstyle\sum_{\varrho=a}^{b-1} \frac{W^{\frac{1}{\beta_1}}(b-1)}{s_1^{\frac{1}{\beta_1}}(\varrho)}\\ &\le \Delta^2 q(a) - W^{\frac{1}{\beta_1}}(b-1)S_1(b,a)\end{aligned}$$

and therefore,

$$\Delta^2 q(a) \ge W^{\frac{1}{\beta_1}}(b-1)S_1(b,a) \tag{38}$$

From (35)-(38), we obtain

$$\begin{aligned}-\Delta W(\vartheta) &\ge k^{\beta_2} s_2(\vartheta) q^{\beta_2}(\vartheta - n + 1)\\ &\ge (\theta_1 k)^{\beta_2} s_2(\vartheta)(\vartheta - n + 1)^{\beta_2}(\Delta q(\vartheta - n))^{\beta_2}\\ &\ge (\theta_1\theta_2 k)^{\beta_2} s_2(\vartheta)(\vartheta - n + 1)^{\beta_2}(\vartheta - n)^{\beta_2}\left(\Delta^2 q(\vartheta - n - 1)\right)^{\beta_2}\\ &\ge (\theta_1\theta_2 k)^{\beta_2} s_2(\vartheta)(\vartheta - n + 1)^{\beta_2}(\vartheta - n)^{\beta_2} W^{\frac{\beta_2}{\beta_1}}(\vartheta - n + k_1 - 2)[S_1(\vartheta - n + k_1 - 1, \vartheta - n - 1)]^{\beta_2}\end{aligned}$$

Therefore, W is a positive and decreasing solution of

$$\Delta W(\vartheta) + (\theta_1\theta_2 k)^{\beta_2} s_2(\vartheta)(\vartheta - n + 1)^{\beta_2}$$
$$(\vartheta - n)^{\beta_2} W^{\frac{\beta_2}{\beta_1}}(\vartheta - n + k_1 - 2)[S_1(\vartheta - n + k_1 - 1, \vartheta - n - 1)]^{\beta_2} \le 0$$

while applying Lemma 2.1(I), (25) also has an eventually positive solution which contradictory.

Case(III) and (IV):

In the rest of the proof, let W be as in (34). Now,

$q(\vartheta) = y(\vartheta) - s_4(\vartheta) y^{\beta_4}(\vartheta - k) \ge -s_4(\vartheta) y^{\beta_4}(\vartheta - k)$ eventually

Thus,

$$y(\vartheta - k) \ge -\left(\frac{q(\vartheta)}{s_4(\vartheta)}\right)^{\frac{1}{\beta_4}} \tag{39}$$

Here from (1) we get eventually,

$$\begin{aligned}-\Delta W(\vartheta) &= s_2(\vartheta) y^{\beta_2}(\vartheta - n + 1) + s_3(\vartheta) y^{\beta_3}(\vartheta + n^*)\\ &\ge s_2(\vartheta) y^{\beta_2}(\vartheta - n + 1)\\ &\ge -\frac{s_2(\vartheta)}{s_4^{\frac{\beta_2}{\beta_4}}(\vartheta - n + k + 1)} q^{\frac{\beta_2}{\beta_4}}(\vartheta - n + k + 1).\end{aligned} \tag{40}$$

Case (III).

From (34) and (28) we obtain,

$$\begin{aligned}\Delta^2 q(\vartheta) &= \Delta^2 q(\vartheta_1) + \sum_{\varrho=\vartheta_1}^{\vartheta-1} \Delta^3 q(\varrho)\\ &= \Delta^2 q(\vartheta_1) - \sum_{\varrho=\vartheta_1} \frac{W^{\frac{1}{\beta_1}}}{s_1^{\frac{1}{\beta_1}}(\varrho)}\\ &\le -\sum_{\varrho=\vartheta_1}^{\vartheta-1} \frac{W^{\frac{1}{\beta_1}}(\vartheta - 1)}{s_1^{\frac{1}{\beta_1}}(\varrho)}\\ &\le -W^{\frac{1}{\beta_1}}(\vartheta - 1) S_1(\vartheta, \vartheta_1)\end{aligned}$$

eventually, which implies from (28) we get

$$\Delta q(\vartheta) = \Delta q(\vartheta_1) + \sum_{\varrho=\vartheta_1}^{\vartheta-1} \Delta^2 q(\varrho)$$
$$\leq \sum_{\varrho=\vartheta_1}^{\vartheta-1} \Delta^2 q(\varrho) \leq -\sum_{\varrho=\vartheta_1}^{\vartheta-1} W^{\frac{1}{\beta_1}}(\vartheta-1) S_1(\varrho,\vartheta_1)$$
$$\leq -W^{\frac{1}{\beta_1}}(\vartheta-2)\sum_{\varrho=\vartheta_1}^{\vartheta-1} S_1(\varrho,\vartheta_1) \text{ eventually}$$

Therefore, from (28) we obtain,

$$q(\vartheta) = q(\vartheta_1) + \sum_{\varrho=\vartheta_1}^{\vartheta-1} \Delta q(\varrho) \leq \sum_{\varrho=\vartheta_1}^{\vartheta-1} \Delta q(\varrho)$$
$$\leq -\sum_{\varrho=\vartheta_1}^{\vartheta-1}\left[W^{\frac{1}{\beta_1}}(\vartheta-2)\left(\sum_{r=\vartheta_1}^{\vartheta-1} S_1(r,\vartheta_1)\right)\right]$$
$$\leq -W^{\frac{1}{\beta_1}}(\vartheta-3)\left[\sum_{\varrho=\vartheta_1}^{\vartheta-1}\left(\sum_{r=\vartheta_1}^{\vartheta-1} S_1(r,\vartheta_1)\right)\right]$$

eventually.

And thus, from (40) we get eventually,

$$-\Delta W(\vartheta) \geq -\frac{s_2(\vartheta)}{s_4^{\frac{\beta_2}{\beta_4}}(\vartheta-n+k+1)} q^{\frac{\beta_2}{\beta_4}}(\vartheta-n+k+1)$$
$$\geq \frac{s_2(\vartheta)}{s_4^{\frac{\beta_2}{\beta_4}}(\vartheta-n+k+1)}\left[W^{\frac{1}{\beta_1}}(\vartheta-n+k-2)\left[\sum_{\varrho=\vartheta_1}^{\vartheta-n+k}\left(\sum_{r=\vartheta_1}^{\vartheta-n+k} S_1(r,\vartheta_1)\right)\right]\right]^{\frac{\beta_2}{\beta_4}}$$
$$\geq \frac{s_2(\vartheta)}{s_4^{\frac{\beta_2}{\beta_4}}(\vartheta-n+k+1)} W^{\frac{\beta_2}{\beta_1}}(\vartheta-n+k-2)\left[\sum_{\varrho=\vartheta_1}^{\vartheta-n+k}\left(\sum_{r=\vartheta_1}^{\vartheta-n+k} S_1(r,\vartheta_1)\right)\right]^{\frac{\beta_2}{\beta_4}}$$

Hence, W is a positive and decreasing solution of

$$\Delta W(\vartheta) + \frac{s_2(\vartheta)}{s_4^{\frac{\beta_2}{\beta_4}}(\vartheta-n+k+1)} W^{\frac{\beta_2}{\beta_1\beta_4}}(\vartheta-n+k-2)\left[\sum_{\varrho=\vartheta_1}^{\vartheta-n+k}\left(\sum_{r=\vartheta_1}^{\vartheta-n+k} S_1(r,\vartheta_1)\right)\right]^{\frac{\beta_2}{\beta_4}} \leq 0$$

while applying Lemma (2.1) I, we see that (26) also has an eventually positive solution, which contradictory.

Case (IV). Let

$a = m-n+k+1, b = a+k_2 > a, c = a+k_3-1 > b-1, d = a+k_4-1 > c-1,$

First, we get eventually,

$$0 \geq q(b) = q(a) + \sum_{\varrho=a}^{b-1} \Delta q(\varrho)$$
$$\geq q(a) + \sum_{\varrho=a}^{b-1} \Delta q(b-1)$$
$$\geq q(a) + (b-a)\Delta q(b-1)$$

$$-q(a) \geq (b-a)\Delta q(b-1) \tag{41}$$

Now, we obtain eventually,

$$0 \leq \Delta q(c) = \Delta q(b-1) + \sum_{\varrho=b-1}^{c-1} \Delta^2 q(\varrho)$$
$$\leq \Delta q(b-1) + (c-b+1)\Delta^2 q(c-1)$$
$$\leq \Delta q(b-1) + c\left(1-\left(\frac{b-1}{c}\right)\right)\Delta^2 q(c-1)$$

Since $\frac{b-1}{c} \to 0$as $c \to \infty$, there exists $\theta_0 \in (0,1)$, such that

$\Delta q(b-1) \geq -c\theta_0 \Delta^2 q(c-1)$ eventually. (42)

Then, from (34) and (28) we have

$$\Delta^2 q(d) = \Delta^2 q(c-1) + \sum_{\varrho=c-1}^{d-1} \Delta^3 q(\varrho)$$
$$\leq -\sum_{\varrho=c-1}^{d-1} \frac{W^{\frac{1}{\beta_1}}(\varrho)}{s_1^{\frac{1}{\beta_1}}(\varrho)}$$
$$\leq -W^{\frac{1}{\beta_1}}(d-1) S_1(d, c-1)$$

$$-\Delta^2 q(d) \geq W^{\frac{1}{\beta_1}}(d-1) S_1(d, c-1) \quad (43)$$

Thus, from (40)-(43) we get

$$-\Delta W(\vartheta) \geq \frac{s_2(\vartheta)}{s_4^{\frac{\beta_2}{\beta_4}}(\vartheta-n+k+1)} [(b-a)\Delta q(b-1)]^{\frac{\beta_2}{\beta_4}}$$
$$\geq \frac{s_2(\vartheta)}{s_4^{\frac{\beta_2}{\beta_4}}(\vartheta-n+k+1)} k_2^{\frac{\beta_2}{\beta_4}} \left[-c\theta_0 \Delta^2 q(c-1)\right]^{\frac{\beta_2}{\beta_4}}$$
$$\geq \frac{s_2(\vartheta)}{s_4^{\frac{\beta_2}{\beta_4}}(\vartheta-n+k+1)} (k_2 c\theta_0)^{\frac{\beta_2}{\beta_4}} \left[W^{\frac{1}{\beta_1}}(c-2) S_1(c-1, c-1)\right]^{\frac{\beta_2}{\beta_4}}$$

which implies W is a positive and decreasing solution of

$$\Delta W(\vartheta) + \frac{s_2(\vartheta)}{s_4^{\frac{\beta_2}{\beta_4}}(\vartheta-n+k+1)} (k_2 c\theta_0)^{\frac{\beta_2}{\beta_4}} \left[W^{\frac{1}{\beta_1}}(c-2) S_1(c-1, c-1)\right]^{\frac{\beta_2}{\beta_4}} \leq 0$$

$$\Delta W(\vartheta) + \frac{s_2(\vartheta)}{s_4^{\frac{\beta_2}{\beta_4}}(\vartheta-n+k+1)} (k_2 c\theta_0)^{\frac{\beta_2}{\beta_4}} \left[W^{\frac{1}{\beta_1}}(\vartheta-n+k+k_3-2) S_1(\vartheta-n+k+k_3-1, \vartheta-n+k+k_3-1)\right]^{\frac{\beta_2}{\beta_4}} \leq 0$$

Using Lemma (2.1) I, (27) also has an eventually positive solution which contradicts.

We provide the next results to demonstrate Theorem 3.1.

Corollary 3.2. Assume that(i)-(iii), (2), (22) and (23) hold. If the first order advanced difference equation (24) and the first oorder delay difference equation (25) and

$$\Delta W(\vartheta) + \min\left\{\frac{s_2(\vartheta)}{s_4^{\frac{\beta_2}{\beta_4}}(\vartheta-n-k+1)} \left[\sum_{\varrho=\vartheta_1}^{\vartheta-n+k} \left(\sum_{r=t_1}^{t-n+k} S_1(r, \vartheta_1)\right)\right]^{\frac{\beta_2}{\beta_4}},\right.$$
$$\left.\frac{s_2(\vartheta)}{s_4^{\frac{\beta_2}{\beta_4}}(\vartheta-n+k+1)} (k_2 c\theta_0)^{\frac{\beta_2}{\beta_4}} [S_1(\vartheta-n+k+k_3-1, \vartheta-n+k+k_3-1)]^{\frac{\beta_2}{\beta_4}}\right\} W^{\frac{\beta_2}{\beta_1\beta_4}}(\vartheta-n+k+k_3-2) = 0$$

(44)

are oscillatory for $1 > \beta_4$ and $\theta_0, \theta_1, \theta_2 \in (0,1)$, then (1) is oscillatory.

Corollary 3.3. Assume $1 > \beta_4$and $\beta_3 \geq \beta_1 \geq \beta_2$. Let(i)-(iii), (2), (22) and (23) hold. If

$$\limsup_{\vartheta\to\infty} \sum_{\varrho=\vartheta-k_0}^{\vartheta-1} \left(\sum_{r=t-k_0}^{\varrho-} \left(\frac{1}{s_1(\varrho)} \sum_{\xi=r-k_0}^{r-1} s_3(\xi)\right)^{\frac{1}{\beta_1}}\right) = \infty, \tag{45}$$

$$\limsup_{\vartheta\to\infty} s_2(\vartheta)(\vartheta-n+1)^{\beta_2}(\vartheta-n)^{\beta_2} S_1[\vartheta-n+k_1-1, \vartheta-n-1]^{\beta_2} = \infty, \tag{46}$$

$$\limsup_{\vartheta\to\infty} \frac{s_2(\vartheta)}{s_4^{\frac{\beta_2}{\beta_4}}(\vartheta-n+k+1)} \left[\sum_{\varrho=\vartheta_1}^{\vartheta-n+k} \left(\sum_{r=t_1}^{t-n+k} S_1(r,\vartheta_1)\right)\right]^{\frac{\beta_2}{\beta_4}} = \infty, \tag{47}$$

and

$$\limsup_{\vartheta\to\infty} \frac{s_2(\vartheta)}{s_4^{\frac{\beta_2}{\beta_4}}(\vartheta-n+k+1)} (k_2 c\theta_0)^{\frac{\beta_2}{\beta_4}} [S_1(\vartheta-n+k+k_3-1, \vartheta-n+k+k_3-1)]^{\frac{\beta_2}{\beta_4}} = \infty, \tag{48}$$

then (1) is oscillatory.

Corollary 3.4.Let $1 \geq \beta_4$, suppose that(i)-(iii), (12) and (22) hold. Assume $\lim_{\vartheta\to\infty} s(\vartheta) = 0$ where

$$S(\vartheta) = -(\beta_4 s_4(\vartheta))^{\frac{-1}{\beta_4}} \tag{49}$$

Let $\theta_0, \theta_1, \theta_2 \in (0,1)$. If (24)-(27) are oscillatory, then so (1).

Corollary 3.5.Let $1 \geq \beta_4$. Assume that(i)-(iii), (2),(22) and (49) hold. Let $\theta_0, \theta_1, \theta_2 \in (0,1)$. if (24), (25) and (44) are oscillatory then (1) is oscillatory.

Corollary 3.6. Let $1 \geq \beta_4$and $\beta_3 \geq \beta_1 \geq \frac{\beta_2}{\beta_4}$. Assume that(i)-(iii), (2),(22) and (49) hold. Let θ_0, $\theta_1, \theta_2 \in (0,1)$. if (45)-(48) hold then (1) is oscillatory.

Examples

Example 4.1. We look at the eqution

$$\Delta\left((\vartheta+2)^3\left(\Delta^3\left(y(\vartheta)-\frac{1}{\vartheta}y^{\frac{1}{3}}(\vartheta-1)\right)\right)\right) = \vartheta^2 y(\vartheta-3) + (\vartheta+3)^4 y^3(\vartheta+6) \tag{50}$$

Now (50) is in the form (1) where $\beta_1 = \beta_3 = 3$, $\beta_2 = 1, \beta_4 = \frac{1}{3} k = 1, n = 4$,

$n^* = 6$, $s_1(\vartheta) = (\vartheta+2)^3, s_2(\vartheta) = \vartheta^2, s_3(\vartheta) = (\vartheta+3)^4, s_4(\vartheta) = \frac{1}{\vartheta}$.

Then (i)-(iii) are fulfilled and so is (2), because

$$S_1(b,a) = \sum_{\varrho=a}^{b-1} \left(\frac{1}{s_1(\varrho)}\right)^{\frac{1}{3}} = \sum_{\varrho=a}^{b-1} \frac{1}{t+2} = \sum_{\varrho=a+2}^{b} \frac{1}{t} \to \infty$$

Now, (see Note 2.4) $k_0 = k_1 = k_2 = 1$and $k_3 = 2$. And thus (22) is fulfilled and we get,

$$\vartheta + n^* - 3k_0 = \vartheta + 3$$
$$\vartheta - n + k_1 - 2 = \vartheta - n + k - 2 = \vartheta - 5$$
$$\vartheta - n + k + k_3 = \vartheta - 3,$$

Furthermore, $\beta_4 < 1$and $\beta_2 < \beta_1 = \beta_3$. So by corollory 3.3 , we choose $s = s_4$ then

$$\begin{aligned} g_1(\vartheta) &= \frac{2}{3}\left(\frac{1}{3}\right)^{\frac{1}{2}} (s(\vartheta))^{\frac{-1}{2}} (s_4(\vartheta))^{\frac{3}{2}} \\ &= \frac{2}{3}\left(\frac{1}{3}\right)^{\frac{1}{2}} \left(\frac{1}{\vartheta}\right)^{\frac{-1}{2}} \left(\frac{1}{\vartheta}\right)^{\frac{3}{2}} \\ &= \frac{2}{3\sqrt{3}}\left(\frac{1}{\vartheta}\right) \to 0 \text{ as } \vartheta \to \infty. \end{aligned}$$

And so (23) is fulfilled. We also compute

$$\sum_{\varrho=\vartheta-k_0}^{\vartheta-1}\left(\sum_{r=t-k_0}^{\varrho-1}\left(\frac{1}{s_1(\varrho)}\sum_{\xi=r-k_0}^{r-1} s_3(\xi)\right)^{\frac{1}{\beta_1}}\right)=\left(\frac{s_3(\vartheta-3)}{s_1(\vartheta-1)}\right)^{\frac{1}{3}}=\vartheta^{\frac{1}{3}}$$

$$s_2(\vartheta)(\vartheta-n+1)^{\beta_2}(\vartheta-n)^{\beta_2}S_1[\vartheta-n+k_1-1,\vartheta-n-1]^{\beta_2}=\vartheta^2(\vartheta-4)$$

$$\frac{s_2(\vartheta)}{s_4^{\frac{\beta_2}{\beta_4}}(\vartheta-n+k+1)}\left[\sum_{\varrho=\vartheta_1}^{\vartheta-n+k}\left(\sum_{r=t_1}^{t-n+k} S_1(r,\vartheta_1)\right)\right]^{\frac{\beta_2}{\beta_4}}=\frac{\vartheta^2\left(1-\frac{2}{\vartheta}\right)^3}{\left(1-\frac{4}{\vartheta}\right)^3}$$

and

$$\frac{s_2(\vartheta)}{s_4^{\frac{\beta_2}{\beta_4}}(\vartheta-n+k+1)}(k_2c\theta_0)^{\frac{\beta_2}{\beta_4}}[S_1(\vartheta-n+k+k_3-1,\vartheta-n+k+k_3-1)]^{\frac{\beta_2}{\beta_4}}=\frac{\vartheta^2\left(1-\frac{2}{\vartheta}\right)^3}{\left(1-\frac{1}{\vartheta}\right)^3}$$

Hence (45)-(48) hold. Now that all of corollary 3.3's criteria have been fulfilled, Equation (50) is oscillatory

Example 4.2.The equations are considered

$$\Delta\left((\vartheta+2)^3\left(\Delta^3\left(y(\vartheta)-\vartheta y^{\frac{1}{3}}(\vartheta-1)\right)=\vartheta^2 y(\vartheta-3)+(\vartheta+3)^4 y^3(\vartheta+6)\right.\right.\qquad(51)$$

Now (51) is in the form (1) where $\beta_1=\beta_3=3, \beta_2=1, \beta_4=\frac{1}{3}$

$k=1, n=4, n^*=6, s_1(\vartheta)=(\vartheta+2)^3, s_2(\vartheta)=\vartheta^2, s_3(\vartheta)=(\vartheta+3)^4, s_4(\vartheta)=l.$ Furthermore, we have $1\geq\beta_4$ and $\beta_3=\beta_1\geq\frac{\beta_2}{\beta_4}$ from corollary 3.6.

We compute $S(\vartheta)=-(\beta_4 s_4(\vartheta))^{-3}=-\left(\frac{\vartheta}{3}\right)^{-3}=-\left(\frac{3}{\vartheta}\right)^3\to 0$, as $l\to\infty$ and so (49) is fulfilled because all of the other constraints of corollary 3.6 are satisfied in the same sense as in Example 4.1, (51) is oscillatory

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