Indian Journal of Natural Sciences www.tnsroindia.org.in ©IJONS

Vol.14 / Issue 80 / Oct / 2023 International Bimonthly (Print) – Open Access ISSN: 0976 – 0997

RESEARCH ARTICLE

Third Order Semi-Canonical Nonlinear Difference Equations

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Received: 16 Aug 2023 Revised: 30 Aug 2023 Accepted: 04 Sep 2023

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ABSTRACT

We provide some novel oscillation conditions for semi-canonical difference equations using the canonical transformation method. Our findings reinforce and improve certain earlier ones. Examples have been generated to highlight the relevance of the consequences.

Keywords: Semi-canonical, asymptotic, oscillatory, difference equation, nonlinear.

INTRODUCTION

Consider the third order difference equation in this instance.

$$\Delta(q(\varsigma)\Delta(r(\varsigma)\Delta\Psi(\varsigma))) = s(\varsigma)\Psi^{\kappa}(\varrho(\varsigma)), \varsigma \geq \varsigma_0 \quad (1.1)$$

where ς_0 is a positive integer. We assume the subsequent hypothesis

H1) $\{q(\varsigma)\}, \{r(\varsigma)\}, \{s(\varsigma)\}$ are positive real sequences;

H2) $\{\varrho(\varsigma)\}$ is an increasing sequence of positive integer with $\varrho(\varsigma) \geq \varsigma + 1$;

H3) κ is a real positive integer with $\kappa > 1$;

H4) the equation (1.1) is in semi canonical form, that is

$$\sum_{\varsigma=\varsigma_0}^{\infty} \frac{1}{q(\varsigma)} = \infty \text{ and } \sum_{\varsigma=\varsigma_0}^{\infty} \frac{1}{r(\varsigma)} < \infty. \quad (1.2)$$

A real sequence $\{\Psi(\varsigma)\}$ that satisfies (1.1) for any $\varsigma \geq \varsigma_0$ is referred to as a solution of (1.1). Nontrivial solutions to (1.1) are either oscillatory or non oscillatory depending on whether the final result is positive or negative.

Finding out the qualitative behavior of various types of second order difference equations has garnered a lot of attention in recent years. Despite the fact that discrete models are used in many other areas of mathematics, including economics, mathematical biology, and [1] and [2], difference equations have a wide range of applications in these fields.

A number of publications have concentrated on the oscillatory and asymptotic solutions of (1.1), see [3]-[13]and the cited references throughout. However, the publication devotes the majority of its space to equation of the canonical form, (i.e),

$$\sum_{\varsigma=\varsigma_0}^{\infty} \frac{1}{q(\varsigma)} = \infty \text{ and } \sum_{\varsigma=\varsigma_0}^{\infty} \frac{1}{r(\varsigma)} = \infty.$$

In [14], the authors studied difference equation of the form

$$\Delta\Big(q(\varsigma)\Delta\big(r(\varsigma)\Delta\Psi(\varsigma)\big)\Big) = s(\varsigma)f\Big(\Psi\big(\varrho(\varsigma)\big)\Big), \qquad \varsigma \geq \varsigma_0$$

with semi-canonical condition

$$\sum_{\varsigma=\varsigma_0}^{\infty} \frac{1}{q(\varsigma)} < \infty \text{ and } \sum_{\varsigma=\varsigma_0}^{\infty} \frac{1}{r(\varsigma)} = \infty.$$

In [15], the authors discussed about the difference equation of the form

$$\Delta(q(\varsigma)\Delta(r(\varsigma)\Delta\Psi(\varsigma))) + w(\varsigma)f(\Psi(\sigma(\varsigma))) - s(\varsigma)g(\Psi(\tau(\varsigma))) = 0.$$

with semi-canonical condition

$$\sum_{\varsigma=\varsigma_0}^{\infty} \frac{1}{q(\varsigma)} < \infty \text{ and } \sum_{\varsigma=\varsigma_0}^{\infty} \frac{1}{r(\varsigma)} = \infty.$$

To the greatest extent of our knowledge, the oscillatory features of the relevant equation have not been studied when

$$\sum_{\varsigma=\varsigma_0}^{\infty} \frac{1}{q(\varsigma)} = \infty \text{ and } \sum_{\varsigma=\varsigma_0}^{\infty} \frac{1}{r(\varsigma)} < \infty.$$

Therefore, the aim of this paper is to provide the qualitative behavior of (1.1) if condition (1.2) is satisfied. This is initially established by transforming semi-canonical to canonical, after which we approach to develop some novel criteria for oscillatory solution of (1.1).

MAIN RESULTS

To make it easier to read, the following symbols will be used:

$$Q(\varsigma) = \sum_{t=\varsigma_1}^{\varsigma-1} \frac{1}{\gamma(t)}, \qquad R(\varsigma) = \sum_{t=\varsigma}^{\infty} \frac{1}{r(t)}, \qquad \xi(\varsigma) = r(\varsigma)R(\varsigma)R(\varsigma+1),$$

$$\gamma(\varsigma) = \frac{q(\varsigma)}{R(\varsigma+1)}, \qquad S(\varsigma) = s(\varsigma)R^{\kappa}\big(\varrho(\varsigma)\big), \qquad \Xi(\varsigma) = \sum_{t=\varsigma_1}^{\varsigma-1} \frac{1}{\xi(t)},$$

$$F(\varsigma) = \frac{1}{\gamma(t)} \sum_{u=\varsigma}^{\infty} S(u), \qquad T(\varsigma) = \sum_{s=\varsigma_1}^{\varsigma-1} \frac{1}{\xi(s)} \sum_{t=\varsigma_1}^{s-1} \frac{1}{\gamma(t)}.$$

where $\varsigma \geq \varsigma_1 \geq \varsigma_0$ and ς_1 is large enough.

Theorem 2.1. Assume that

$$\sum_{\varsigma=\varsigma_0}^{\infty} \frac{1}{\gamma(\varsigma)} = \infty \qquad (2.1)$$

Then (1.1) has the following canonical representation

$$\Delta\left(\frac{q(\varsigma)}{R(\varsigma+1)}\Delta\left(r(\varsigma)R(\varsigma)R(\varsigma+1)\Delta\left(\frac{\Psi(\varsigma)}{R(\varsigma)}\right)\right)\right) = \Delta(q(\varsigma)\Delta(r(\varsigma)\Delta\Psi(\varsigma))). \qquad (2.2)$$

Proof. Direct computation demonstrates that

$$\begin{aligned}\frac{q(\varsigma)}{R(\varsigma+1)}\Delta\left(r(\varsigma)R(\varsigma)R(\varsigma+1)\Delta\left(\frac{\Psi(\varsigma)}{R(\varsigma)}\right)\right) &= \frac{q(\varsigma)}{R(\varsigma+1)}\Delta(r(\varsigma)R(\varsigma)\Delta\Psi(\varsigma) + \Psi(\varsigma)) \\ &= \frac{q(\varsigma)}{R(\varsigma+1)}[R(\varsigma+1)\Delta(r(\varsigma)\Delta\Psi(\varsigma))] \\ &= q(\varsigma)\Delta(r(\varsigma)\Delta\Psi(\varsigma))\end{aligned}$$

Hence

$$\Delta\left(\frac{q(\varsigma)}{R(\varsigma+1)}\Delta\left(r(\varsigma)R(\varsigma)R(\varsigma+1)\Delta\left(\frac{\Psi(\varsigma)}{R(\varsigma)}\right)\right)\right) = \Delta(q(\varsigma)\Delta(r(\varsigma)\Delta\Psi(\varsigma))).$$

As we can see from (2.1),

$\sum_{\varsigma=\varsigma_0}^{\infty} \frac{R(\varsigma+1)}{q(\varsigma)} = \infty,$ (2.3)

and since

$$\sum_{\varsigma=\varsigma_0}^{\infty} \frac{1}{r(\varsigma)R(\varsigma)R(\varsigma+1)} = \sum_{\varsigma=\varsigma_0}^{\infty} \Delta\left(\frac{1}{R(\varsigma)}\right) = \lim_{\varsigma\to\infty} \frac{1}{R(\varsigma)} - \frac{1}{R(\varsigma_0)} = \infty,$$

we assess that (2.2) is in canonical form

Corollary 2.2. Assume that (2.1) holds. Then the semi-canonical difference equation (1.1) possesses a solution $\Psi(\varsigma)$ if and only if the canonical equation

$\Delta(\gamma(\varsigma)\Delta(\xi(\varsigma)\Delta\Phi(\varsigma))) = S(\varsigma)\Phi^{\kappa}(\varrho(\varsigma))$ (2.4)

has the positive solution $\Phi(\varsigma) = \frac{\Psi(\varsigma)}{R(\varsigma)}$.

In the following section, we provide the structure of a potential non-oscillatory solution to (2.4), which is deduced from an analogy involving the discrete knesers theorem and canonical form of (2.4)

$\Phi(\varsigma) \in \mathcal{N}_0 : \Phi(\varsigma) > 0, \Delta\Phi(\varsigma) > 0, \Delta(\xi(\varsigma)\Delta(\Phi(\varsigma))) < 0, \Delta(\gamma(\varsigma)\Delta(\xi(\varsigma)\Delta\Phi(\varsigma))) > 0$

and

$\Phi(\varsigma) \in \mathcal{N}_3 : \Phi(\varsigma) > 0, \Delta\Phi(\varsigma) > 0, \Delta(\xi(\varsigma)\Delta(\Phi(\varsigma))) > 0, \Delta(\gamma(\varsigma)\Delta(\xi(\varsigma)\Delta\Phi(\varsigma))) > 0.$

Lemma 2.3**.** Assuming $\Phi(\varsigma) \in \mathcal{N}_3$ is a positive solution of (2.4) and

$\sum_{u=\varsigma_1}^{\infty} S(u)T^{\kappa}(\varrho(u)) = \infty.$ (2.5)

Then $\frac{\Phi(\varsigma)}{T(\varsigma)}$ is eventually increasing $\forall\ \varsigma \geq K$.

The preceding lemma's proof resembles that of in [3]**.**

Theorem 2.4. Assuming that (2.1) is true,

$\sum_{t=\varsigma_1}^{\varsigma-1} \frac{1}{\xi(t)} \sum_{u=t}^{\infty} F(t) = \infty$ (2.6)

and

$\limsup_{\varsigma\to\infty} \left(\frac{1}{\Xi^{\kappa}(\varrho(\varsigma))} \sum_{t=\varsigma_1}^{\varsigma-1} \Xi^{\kappa}(\varrho(t))\Xi(t+1)F(t) + \sum_{t=\varsigma}^{\varrho(t)-1} \Xi(t+1)F(t) + \Xi(\varrho(\varsigma)) \sum_{t=\varrho(\varsigma)}^{\infty} F(t)\right) = \infty.$ (2.7)

Then all non-oscillatory solution of (1.1) meets $\lim_{\varsigma\to\infty} \frac{\Psi(\varsigma)}{R(\varsigma)} = 0$.

Proof. Let $\{\Psi(\varsigma)\}$ be an non-oscillatory solution (1.1). Without losing generality, suppose that $\{\Psi(\varsigma)\}$ is a positive solution of (1.1). Then, from corollary 2.2, relevant sequence $\{\Phi(\varsigma)\}$ is also a positive solution of (2.4) and $\Phi(\varsigma) \in \mathcal{N}_0$ or $\Phi(\varsigma) \in \mathcal{N}_3$ for all $\varsigma \geq \varsigma_1$. Now consider that $\Phi(\varsigma) \in \mathcal{N}_0$ for all $\varsigma \geq \varsigma_1$. From the monotonicity of $\xi(\varsigma)\Delta\Phi(\varsigma)$, we carry

$$\Phi(\varsigma) \geq \Phi(\varsigma) - \Phi(\varsigma_1) = \sum_{t=\varsigma_1}^{\varsigma-1} \frac{\xi(t)\Delta\Phi(t)}{\xi(t)} \geq \xi(\varsigma)\Delta\Phi(\varsigma)\Xi(\varsigma)$$

which suggest that

$\Delta\left(\frac{\Phi(\varsigma)}{\Xi(\varsigma)}\right) = \left(\frac{\Xi(\varsigma)\Delta\Phi(\varsigma) - \Phi(\varsigma)\frac{1}{\xi(\varsigma)}}{\Xi(\varsigma)\Xi(\varsigma+1)}\right) \leq 0$ (2.8)

Hence $\frac{\Phi(\varsigma)}{\Xi(\varsigma)}$ is decreasing.

On the other hand, summing twice (2.4) from ς to ∞ gives

$$\begin{aligned} \xi(\varsigma)\Delta\Phi(\varsigma) &\geq \sum_{t=\varsigma}^{\infty} \frac{\Phi^{\kappa}(\varrho(t))}{\gamma(t)} \sum_{u=t}^{\infty} S(u) \\ &\geq \sum_{t=\varsigma}^{\infty} \Phi^{\kappa}(\varrho(t))F(t). \end{aligned}$$

Again, summing from ς_1 to $\varsigma - 1$, we carry

$\Phi(\varsigma) \geq \sum_{t=\varsigma_1}^{\varsigma-1} \frac{1}{\xi(t)} \sum_{u=\varsigma}^{\infty} \Phi^{\kappa}(\varrho(u))F(u)$
$= \Xi(\varsigma)\sum_{u=\varsigma}^{\infty} \Phi^{\kappa}(\varrho(u))F(u) + \sum_{t=\varsigma_1}^{\varsigma-1} \Xi(t+1)\Phi^{\kappa}(\varrho(t))F(t)$

Hence

$$\Phi(\varrho(\varsigma)) \geq \sum_{t=\varsigma_1}^{\varsigma-1} \Xi(t+1)\Phi^{\kappa}(\varrho(t))F(t) + \sum_{t=\varsigma}^{\varrho(\varsigma)-1} \Xi(t+1)\Phi^{\kappa}(\varrho(t))F(t) + \Xi(\varrho(\varsigma)) \sum_{t=\varrho(\varsigma)}^{\infty} \Phi^{\kappa}(\varrho(t))F(t).$$

Using $\Phi(\varsigma)$ is increasing and $\frac{\Phi(\varsigma)}{\Xi(\varsigma)}$ decreasing, we obtain

$$\Phi(\varrho(\varsigma)) \geq \frac{\Phi^{\kappa}(\varrho(\varsigma))}{\Xi^{\kappa}(\varrho(\varsigma))} \sum_{t=\varsigma_1}^{\varsigma-1} \Xi^{\kappa}(\varrho(t))\Xi(t+1)F(t) + \Phi^{\kappa}(\varrho(\varsigma)) \sum_{t=\varsigma}^{\varrho(\varsigma)-1} \Xi(t+1)F(t) + \Phi^{\kappa}(\varrho(\varsigma))\Xi(\varrho(\varsigma)) \sum_{t=\varrho(\varsigma)}^{\infty} F(t)$$

or

$$\Phi^{1-\kappa}(\varrho(\varsigma)) \geq \frac{1}{\Xi^{\kappa}(\varrho(\varsigma))}\sum_{t=\varsigma_1}^{\varsigma-1} \Xi^{\kappa}(\varrho(t))\Xi(t+1)F(t) + \sum_{t=\varsigma}^{\varrho(\varsigma)-1} \Xi(t+1)F(t) + \Xi(\varrho(\varsigma))\sum_{t=\varrho(\varsigma)}^{\infty} F(t).$$

(2.9)

Since $\Phi(\varsigma)$ is positive and increasing, ∃ a constant $A > 0 \ni \Phi(\varsigma) \geq A$, and so we carry $\Phi^{1-\kappa}(\varrho(\varsigma)) \leq A^{1-\kappa}$. Using this in (2.9) we obtain

$$A^{1-\kappa} \geq \frac{1}{\Xi^{\kappa}(\varrho(\varsigma))} \sum_{t=\varsigma_1}^{\varsigma-1} \Xi^{\kappa}(\varrho(t))\Xi(t+1)F(t) + \sum_{t=\varsigma}^{\varrho(\varsigma)-1} \Xi(t+1)F(t) + \Xi(\varrho(\varsigma)) \sum_{t=\varrho(\varsigma)}^{\infty} F(t).$$

Taking limsup as $\varsigma \to \infty$ of the aforementioned inequality, which contradicts (2.7)

Next, we assume that $\Phi(\varsigma) \in \mathcal{N}_3$. Since $\Phi(\varsigma)$ is positive and increasing, there exists$\lim_{\varsigma\to\infty}\Phi(\varsigma) = e \geq 0$. Suppose that $e > 0$, then $\Phi(\varsigma) \geq e > 0$. Summing twice (2.4) from ς to ∞ and again summing from ς_1 to $\varsigma - 1$, which gives

$\Phi(\varsigma) \geq e^{\kappa} \sum_{t=\varsigma_1}^{\varsigma-1} \frac{1}{\xi(t)} \sum_{u=t}^{\infty} F(t),$

which contradicts (2.6) and so

$\lim_{\varsigma\to\infty}\Phi(\varsigma) = \lim_{\varsigma\to\infty} \frac{\Psi(\varsigma)}{R(\varsigma)} = 0.$

Hence the proof is completed.

Theorem 2.5. Let (2.1) and (2.5) hold, and

$\lim_{\varsigma\to\infty} w^{\frac{1}{\kappa}-1}(\varsigma) = M_1^{\frac{1}{\kappa}-1} < \infty.$ (2.10)

Suppose

$\limsup_{\varsigma\to\infty} \left\{\frac{1}{T(\varrho(\varsigma))}\sum_{t=\varsigma}^{\varrho(\varsigma)-1} \frac{1}{\xi(t)}\sum_{u=\varsigma}^{t-1} \frac{1}{\gamma(u)}\sum_{v=\varsigma}^{u-1} S(v)T^{\kappa}(\varrho(v))\right\} > M_1,$ (2.11)

then $\mathcal{N}_3$ is $\emptyset$ for (1.1).

Proof. Let $\{\Psi(\varsigma)\}$ be an positive solution of (1.1). By Corollary 2.2 the corresponding function $\{\Phi(\varsigma)\}$ is a positive solution of (2.4). Let us assume that $\Phi(\varsigma) \in \mathcal{N}_3$ for all $\varsigma \geq \varsigma_1$. Summing (2.4) from ς to $m - 1$, we have

$$\begin{aligned}\Delta(\xi(m)\Delta\Phi(m)) &\geq \frac{1}{\gamma(m)} \sum_{t=\varsigma}^{m-1} S(t)\Phi^{\kappa}(\varrho(t)) \\ &\geq \frac{1}{\gamma(m)} \sum_{t=\varsigma}^{m-1} S(t)\frac{\Phi^{\kappa}(\varrho(t))}{T^{\kappa}(\varrho(t))}T^{\kappa}(\varrho(t)) \\ &\geq \frac{\Phi^{\kappa}(\varrho(\varsigma))}{T^{\kappa}(\varrho(\varsigma))}\frac{1}{\gamma(m)} \sum_{t=\varsigma}^{m-1} S(t)T^{\kappa}(\varrho(t)).\end{aligned}$$

Again summing, we get

$$\Delta\Phi(m) \geq \frac{\Phi^{\kappa}(\varrho(\varsigma))}{T^{\kappa}(\varrho(\varsigma))}\frac{1}{\xi(m)} \sum_{t=\varsigma}^{m-1} \frac{1}{\gamma(t)} \sum_{u=\varsigma}^{t-1} S(u)T^{\kappa}(\varrho(u)).$$

Summing once again, we obtain

$\Phi(m) \geq \frac{\Phi^{\kappa}(\varrho(\varsigma))}{T^{\kappa}(\varrho(\varsigma))}\sum_{t=\varsigma}^{m-1} \frac{1}{\xi(t)}\sum_{u=\varsigma}^{t-1} \frac{1}{\gamma(u)}\sum_{v=\varsigma}^{u-1} S(v)T^{\kappa}(\varrho(v))$.

Now, we have to define $m = \varrho(\varsigma)$ and $w(\varsigma) = \frac{\Phi^{\kappa}(\varrho(\varsigma))}{T^{\kappa}(\varrho(\varsigma))}$.

$$\Phi(\varrho(\varsigma)) \geq w(\varsigma)\sum_{t=\varsigma}^{\varrho(\varsigma)-1} \frac{1}{\xi(t)}\sum_{u=\varsigma}^{t-1} \frac{1}{\gamma(u)}\sum_{v=\varsigma}^{u-1} S(v)T^{\kappa}(\varrho(v)),$$

$w^{\frac{1}{\kappa}-1}(\varsigma) \geq \frac{1}{T(\varrho(\varsigma))}\sum_{t=\varsigma}^{\varrho(\varsigma)-1} \frac{1}{\xi(t)}\sum_{u=\varsigma}^{t-1} \frac{1}{\gamma(u)}\sum_{v=\varsigma}^{u-1} S(v)T^{\kappa}(\varrho(v))$.

Taking limit as $\varsigma \to \infty$ on above inequality, we get

$$\limsup_{\varsigma\to\infty} w^{\frac{1}{\kappa}-1}(\varsigma) \geq \limsup_{\varsigma\to\infty}\frac{1}{T(\varrho(\varsigma))}\sum_{t=\varsigma}^{\varrho(\varsigma)-1} \frac{1}{\xi(t)}\sum_{u=\varsigma}^{t-1} \frac{1}{\gamma(u)}\sum_{v=\varsigma}^{u-1} S(v)T^{\kappa}(\varrho(v)).$$

Which contradicts (2.11). Thus, $\mathcal{N}_3$ is $\emptyset$ for (1.1). This completes the proof.

Theorem 2.6. If all the conditions of Theorem 2.4 and 2.5 hold, then the solution $\Psi(\varsigma)$ of (1.1) is oscillatory or satisfies $\lim_{\varsigma\to\infty}\frac{\Psi(\varsigma)}{R(\varsigma)} = 0$

3. EXAMPLES

Example 3.1**.** Take

$$\Delta\left(\frac{1}{(2^{\varsigma+1})^2}\Delta\left(\frac{1}{2^{\varsigma+1}}\Delta\Psi(\varsigma)\right)\right) = (2^{\varsigma+2})^2\Psi^3(\varsigma+2), \varsigma \geq 1 \qquad (3.1)$$

Here $q(\varsigma) = \frac{1}{(2^{\varsigma+1})^2}, r(\varsigma) = \frac{1}{2^{\varsigma+1}}, s(\varsigma) = (2^{\varsigma+1})^2, \kappa = 3, \varrho(\varsigma) = \varsigma + 2, R(\varsigma) = \sum_{t=\varsigma}^{\infty} 2^{t+1} = 2^{\varsigma}, \gamma(\varsigma) = \frac{1}{(2^{\varsigma+1})^3}, \xi(\varsigma) = 2^{\varsigma}, S(\varsigma) = (2^{\varsigma+2})^5, Q(\varsigma) \approx (2^{\varsigma+2})^3, T(\varsigma) \approx 2^{\varsigma}(2^{\varsigma+1})^3, \Xi(\varsigma) = \frac{1}{2^{\varsigma}}, F(\varsigma) = (2^{\varsigma+1})^2$. Therefore (2.4) becomes

$$\Delta\left(\frac{1}{(2^{\varsigma+1}+1)^3}\Delta(2^{\varsigma}\Delta\Phi(\varsigma))\right) = (2^{\varsigma+2})^5\Phi^3(\varsigma+2).$$

is canonical.

Clearly

$$\sum_{\varsigma=\varsigma_0}^{\infty}\frac{1}{\gamma(\varsigma)} = \sum_{\varsigma=1}^{\infty}(2^{\varsigma+1})^2 = \infty$$

and

$$\sum_{t=\varsigma_1}^{\varsigma-1}\frac{1}{2^t}\sum_{u=t}^{\infty}2^{u+1} = \infty$$

Also (2.7) becomes

$$\limsup_{\varsigma\to\infty}\left(2^{\varsigma+2}\sum_{t=1}^{\varsigma-1}\frac{1}{2^{t+2}}\frac{1}{2^{t+1}}(2^{t+1})^2 + \sum_{t=\varsigma}^{\varsigma+1}(2^{t+1}) + \frac{1}{2^{\varsigma+1}}\sum_{t=\varsigma+2}^{\infty}(2^{t+1})^2\right) = \infty.$$

(i.e), (2.7) is verified. Hence by Theorem 2.4, every non-oscillatory solution $\Psi(t)$ of (3.1) satisfies $\lim_{\varsigma\to\infty}\frac{1}{2^{\varsigma}}\Psi(\varsigma) = 0$.

Example 3.2. Take

$$\Delta\left(\frac{1}{\varsigma^2}\Delta(\varsigma(\varsigma+1)\Delta\Psi(\varsigma))\right) = \varsigma(\varsigma+2)^2\Psi^2(\varsigma+1). \qquad (3.2)$$

Here $q(\varsigma) = \frac{1}{\varsigma^2}, r(\varsigma) = \varsigma(\varsigma+1), s(\varsigma) = \varsigma(\varsigma+2)^2, \kappa = 2, \varrho(\varsigma) = \varsigma + 1, R(\varsigma) = \sum_{t=\varsigma}^{\infty}\frac{1}{t(t+1)} = \frac{1}{\varsigma}, \quad \gamma(\varsigma) = \frac{1}{\varsigma}, \xi(\varsigma) = 1, S(\varsigma) = \varsigma(\varsigma+2)^2\frac{1}{(\varsigma+1)^2}, Q(\varsigma) \approx \varsigma, T(\varsigma) \approx \varsigma^2, \Xi(\varsigma) \approx \varsigma$. Take $M = 1$, therefore (2.4) becomes

$$\Delta\left(\frac{1}{\varsigma}(\Delta(\Delta\Phi(\varsigma)))\right) = \varsigma(\varsigma+2)^2\frac{1}{(\varsigma+1)^2}\Phi^2(\varsigma+1),$$

is canonical.

Clearly

$\sum_{\varsigma=\varsigma_0}^{\infty}\frac{1}{\gamma(\varsigma)} = \sum_{\varsigma=1}^{\infty}\varsigma = \infty$

and

$$\sum_{u=\varsigma_1}^{\infty} S(u)T^{\kappa}(\varrho(u)) = \sum_{u=\varsigma_1}^{\infty} u(u+2)^2 = \infty$$

and

$$\lim_{\varsigma\to\infty} w^{\frac{1}{\varsigma}-1}(\varsigma) = 1 < \infty$$

is true.

Also (2.11) becomes

$$\limsup_{\varsigma\to\infty}\left\{\frac{1}{\varsigma+1}\sum_{t=1}^{\varsigma}\sum_{u=1}^{t-1}\frac{1}{\varsigma}\sum_{v=1}^{u-1} v(v+2)^2\right\} = \infty > M_1.$$

Hence all the conditions of Theorem 2.5 are verified, so every solution $\Psi(\varsigma)$ of (3.2) is oscillatory.

CONCLUSION

This study finds some fresh oscillatory and asymptotic conditions for semi-canonical difference equations. The conclusions reached in this study have a high degree of generality, and they enhance and add to prior conclusions for specific instances of equation (1.1).

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