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27th October 2021

Jointly Organized by

Department of Biological Science, Physical Science and Computational Science

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ABOUT THE INSTITUTION

A nations's growth is in proportion to education and intelligence spread among the masses. Having this idealistic vision, two great philanthropists late. S.P. Nallamuthu Gounder and Late. Arutchelver Padmabhushan Dr.N.Mahalingam formed an organization called Pollachi Kalvi Kazhagam, which started NGM College in 1957, to impart holistic education with an objective to cater to the higher educational needs of those who wish to aspire for excellence in knowledge and values. The College has achieved greater academic distinctions with the introduction of autonomous system from the academic year 1987-88. The college has been Re-Accredited by NAAC and it is ISO 9001 : 2015 Certified Institution. The total student strength is around 6000. Having celebrated its Diamond Jubilee in 2017, the college has blossomed into a premier Post-Graduate and Research Institution, offering 26 UG, 12 PG, 13 M.Phil and 10 Ph.D Programmes, apart from Diploma and Certificate Courses. The college has been ranked within Top 100 (72nd Rank) in India by NIRF 2021.

ABOUT CONFERENCE

The International conference on “Emerging Trends in Science and Technology (ETIST-2021)” is being jointly organized by Departments of Biological Science, Physical Science and Computational Science - Nallamuthu Gounder Mahalingam College, Pollachi along with ISTE, CSI, IETE, IEE & RIYASA LABS on 27th OCT 2021. The Conference will provide common platform for faculties, research scholars, industrialists to exchange and discus the innovative ideas and will promote to work in interdisciplinary mode.

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Oscillation of Third Order Difference Equations with Bounded and Unbounded Neutral Coefficients

S. Kaleeswari[1] and **Said. R. Grace**[2]

Abstract - This paper aims the oscillatory behavior of solutions to a class of third order difference equations with bounded and unbounded neutral coefficients. New oscillation results for all solutions to be oscillatory are obtained. Examples are provided to illustrate the main results.

Keywords *Bounded; difference equations; neutral terms; nonlinear; oscillation; unbounded.*
2010 Subject classification: *39A10, 39A21.*

1 Introduction

In this paper, we are concerned with the oscillation of all solutions of the third order difference equations with bounded and unbounded neutral coefficients of the form

$$\Delta^3 \left(y\left(n\right)+p\left(n\right)y\left(\tau\left(n\right)\right)\right)+q\left(n\right)y^{\alpha}\left(\sigma\left(n\right)\right)=0, n\geq n_0 \tag{1}$$

where $n \in N\left(n_0\right)=\left\{n_0, n_0+1, ...\right\}, n_0$ is a positive integer. We use the following assumptions throught the paper.

(H1) $\{p\left(n\right)\}$ is positive real sequence with $p\left(n\right)\geq 1, p\left(n\right)$ not identically one for large n and $\{q\left(n\right)\}$ is nonnegative real sequence and does not vanish eventually;

(H2) α is a ratio of odd positive integers;

(H3) $\{\tau\left(n\right)\}$ and $\{\sigma\left(n\right)\}$ are strictly increasing sequences of integers with $\tau\left(n\right)<n$ with $\lim\limits_{n\to\infty}\tau\left(n\right)=\infty$ and $\sigma\left(n\right)<n$ with $\lim\limits_{n\to\infty}\sigma\left(n\right)=\infty$;

(H4) there exists a constant u with $0<u\leq 1$ and

$$\left(\frac{n}{\tau\left(n\right)}\right)^{\frac{2}{u}}\frac{1}{p\left(n\right)}\leq 1 \tag{2}$$

[1]Department of Mathematics, Nallamuthu Gounder Mahalingam College, Pollachi-642001, Coimbatore, Tamilnadu, India.
E.mail: kaleesdesika@gmail.com
[2]Department of Engineering Mathematics, Faculty of Engineering, Cairo University, Orman, Giza 12221, Egypt.
E-mail: saidgrace@yahoo.com

Let $\theta = \min\{\tau(n), \sigma(n)\}$. By a solution of (1), we mean a sequence $\{y(n)\}$ defined for all $n \geq \theta$ and satisfying (1) for all $n \in N$. We consider only solutions of (1) that satisfy $\sup\{|y(n)| : n \geq N\} > 0$ for all $N \geq n_0$ and we tacitly assume that (1) possesses such solutions. A solution of (1) is called oscillatory if it is neither eventually positive nor eventually negative, and otherwise it is called nonoscillatory.

The qualitative analysis of solutions to various classes of third and higher order neutral difference equations have been attracting attention of researchers in recent years, see the monographs [1, 2] and we mention the papers [3-16, 21-25] and the references cited therein. Functional difference equations have many applications in engineering and natural sciences. For instance, neutral type difference equations have been applied to problems in economics, mathematical biology, image analysis and many other areas(see [17-20]).

The above cited papers except [12] were concerned with the case where $p(n)$ is bounded, and so the results obtained in these papers cannot be applied to the case $p(n) \to \infty$ as $n \to \infty$. Based on this observation, the aim of this paper is to obtain some new oscillation criteria that can be applied not only to the case where $p(n)$ is unbounded but also to the case where $p(n)$ is bounded. The results established here are motivated by the oscillation results of [7-10].

Without loss of generality, we deal only with positive solutions of (1); since $y(n)$ is a solution of (1), then $-y(n)$ is also a solution.

2 Main Results

To obtain the main results, we shall use the following notations.
For all large $n \geq n_0 > 0$, we define

$$z(n) = y(n) + p(n)y(\tau(n)), h(n) = \tau^{-1}(\sigma(n)), g(n) = \tau^{-1}(\eta(n)),$$

$$\Pi_1(n) = \frac{1}{p(\tau^{-1}(n))}\left[1 - \left(\frac{\tau^{-1}(\tau^{-1}(n))}{\tau^{-1}(n)}\right)^{\frac{2}{u}} \frac{1}{p(\tau^{-1}(\tau^{-1}(n))}\right]$$

and

$$\Pi_2(n) = \frac{1}{p(\tau^{-1}(n))}\left[1 - \frac{1}{p(\tau^{-1}(\tau^{-1}(n))}\right],$$

where $\{\eta(n)\}$ is realvalued positive sequence. We can notice that the sequences $\{\Pi_1(n)\}$ and $\{\Pi_2(n)\}$ are nonnegative because of the condition (2).

Lemma 2.1. *If the sequence $\{h(n)\}$ is such that $\Delta^i h(n) > 0, i = 0, 1, 2, ...m$ and $\Delta^{m+1}h(n) \leq 0, \Delta^{m+1}h(n)$ does not vanish eventually for $n \geq N$, then for every $0 < u \leq 1$, we have*

$$\frac{h(n)}{\Delta h(n)} \geq u\frac{n}{m}$$

eventually.

Proof. By monotonicity of $\Delta^i h(n)$, for any $0 < u \leq 1$, we have

$$\Delta^{i-1}h(n) > \sum_{s=n_0}^{n-1} \Delta^i h(s) \geq (n-n_0)\Delta^i h(n) \geq un\Delta^i h(n).$$

Define the sequence $\{\rho_i(n)\}$, $i = 1, 2, ..., m$ as follows:

$$\rho_1(n) = \Delta^{i-1}h(n) - un\Delta^i h(n)$$

$$\rho_2(n) = 2\Delta^{i-2}h(n) - un\Delta^{i-1}h(n)$$

....

....

$$\rho_i(n) = ih(n) - un\Delta h(n)$$

Clearly $\rho_i(n) > 0$ eventually for $i = 1, 2, ..., m$.
Thus $mh(n) > un\Delta h(n)$, which implies

$$\frac{h(n)}{\Delta h(n)} > u\frac{n}{m}.$$

This completes the proof of the lemma. □

Lemma 2.2. *For $n_1 \geq n_0$, assume that $y(n)$ is an eventually positive solution of (1). Then $z(n)$ satisfies one of the following two cases for $n_2 \geq n_1$.*

(I) $z(n) > 0, \Delta z(n) > 0, \Delta^2 z(n) > 0, \Delta^3 z(n) \leq 0,$

(II) $z(n) > 0, \Delta z(n) < 0, \Delta^2 z(n) > 0, \Delta^3 z(n) \leq 0$

for $n \geq n_2$.

Proof. Since the proof is immediate, it is omitted. □

Lemma 2.3. *Suppose that $y(n)$ is an eventually positive solution of (1) and $z(n)$ satisfies case (I) of Lemma 2.2 for $n \geq n_2$ for some $n_2 \geq n_1$. Then there exists a $n_u \geq n_2$ for every $0 < u \leq 1$ such that*

$$\Delta\left(\frac{z(n)}{n^{\frac{2}{u}}}\right) \leq 0, \tag{3}$$

for $n \geq n_u$.

Proof. Suppose that $z(n)$ satisfies case (I) of Lemma 2.2 for $n \geq n_2$ for some $n_2 \geq n_1$. Then by Lemma 2.1, there exists a $n_u \geq n_2$ for every $0 < u \leq 1$ such that

$$z(n) \geq \frac{u}{2} n \Delta z(n) \, (for) \, n \geq n_u \tag{3*}$$

From (3*), we have

$$\Delta \left(\frac{z(n)}{n^{\frac{2}{u}}} \right) = \frac{n^{2/u} \Delta z(n) - z(n) \Delta n^{2/u}}{n^{2/u}(n+1)^{2/u}} \leq \frac{\Delta z(n)}{(n+1)^{2/u}} - \frac{z(n)}{n^{2/u}} \leq 0,$$

for $n \geq n_u$. This completes the proof of the lemma. □

Lemma 2.4. *Suppose that $y(n)$ is eventually positive solution of (1) with $z(n)$ satisfying case (I) of Lemma 2.2. If*

$$\sum_{u=n_0}^{u=\infty} \sum_{s=u}^{s=\infty} q(s) \Pi_1^{\alpha}(\sigma(s)) h^{\alpha}(s) = \infty, \tag{4}$$

then

(i) z satisfies the inequality

$$\Delta^3 z(n) + q(n) \Pi_1^{\alpha}(\sigma(n)) z^{\alpha}(h(n)) \leq 0 \tag{5}$$

for large n;

(ii) $\Delta z(n) \to \infty$ as $n \to \infty$;

(iii) $z(n)/n$ is increasing.

Proof. Assume that $y(n)$ is an eventually positive solution of (1) such that $y(n) > 0, y(\tau(n)) > 0$ and $y(\sigma(n) > 0$ for $n \geq n_1$ for some $n_1 \geq n_0$. From the definition of z, we have

$$\begin{aligned} y(n) &= \frac{1}{p(\tau^{-1}(n))} \left[z(\tau^{-1}(n)) - y(\tau^{-1}(n) \right] \\ &\geq \frac{z(\tau^{-1}(n))}{p(\tau^{-1}(n))} - \frac{1}{p(\tau^{-1}(n)) p(\tau^{-1}(\tau^{-1}(n)))} z(\tau^{-1}(\tau^{-1}(n))) \end{aligned} \tag{6}$$

Since $\tau(n) < n$ and τ is strictly increasing, we have τ^{-1} is increasing and $n < \tau^{-1}(n)$. Thus

$$\tau^{-1}(n) \leq \tau^{-1}(\tau^{-1}(n)). \tag{7}$$

Now $z(n)$ satisfies case (I) for $n \geq n_2$, by Lemma 2.3, there exists a $n_u \geq n_2$ such that (3) holds for $n \geq n_u$. From (3) and (7), we obtain

$$z(\tau^{-1}(\tau^{-1}(n))) \leq \frac{(\tau^{-1}(\tau^{-1}(n)))^{2/u} z(\tau^{-1}(n))}{(\tau^{-1}(n))^{2/u}}. \tag{8}$$

Use (8) in (6) to get

$$y(n) \geq \Pi_1(n)z(\tau^{-1}(n)) \text{ for } n \geq n_u. \tag{9}$$

Since $\lim_{n\to\infty} \sigma(n) = \infty$, there exists a $n_3 \geq n_u$ such that $\sigma(n) \geq n_u$ for all $n \geq n_3$. Thus it follows from (9) that

$$x(\sigma(n)) \geq \Pi_1(\sigma(n))z(\tau^{-1}(\sigma(n))) \text{ for } n \geq n_3. \tag{10}$$

Substituting (10) in (1) yields

$$\Delta^3 z(n) + q(n)\Pi_1^\alpha(\sigma(n))z^\alpha(h(n)) \leq 0 \text{ for } n \geq n_3. \tag{11}$$

Thus (5) holds.

Next we have to claim that equation (4) implies $\Delta z(n) \to \infty$ as $n \to \infty$. Suppose that $\Delta z(n)$ does not tend to ∞ as $n \to \infty$, which implies that there exists a constant $k > 0$ such that $\lim_{n\to\infty} \Delta z(n) = k$ and so $\Delta z(n) \leq k$. Since $\Delta z(n)$ is positive and increasing for $n \geq n_2$, there exists $n_3 \geq n_2$ and a constant $c > 0$ such that

$$\Delta z(n) \geq c \text{ for } n \geq n_3.$$

This implies

$$z(n) \geq cn \text{ for } n \geq n_4,$$

for some $n_4 \geq n_3$ and some $c > 0$. Since $\lim_{n\to\infty} h(n) = \infty$, we can choose $n_5 \geq n_4$ such that $h(n) \geq n_4$ for $n \geq n_5$. Therefore,

$$z(h(n)) \geq ch(n).$$

Using this in (11) yields

$$\Delta^3 z(n) + c^\alpha q(n)\Pi_1^\alpha(\sigma(n))h^\alpha(n) \leq 0 \text{ for } n \geq n_5.$$

Summing this inequality from n to ∞, we obtain

$$\Delta^2 z(n) \geq c^\alpha \sum_{s=n}^{\infty} q(s)\Pi_1^\alpha(\sigma(s))z^\alpha(h(s))$$

Again summing from n_5 to $n-1$ gives

$$k \geq \Delta z(n) \geq c^\alpha \sum_{u=n_5}^{n-1} \sum_{s=u}^{\infty} q(s)\Pi_1^\alpha(\sigma(s))z^\alpha(h(s)),$$

which is a contradiction to (4) and hence the claim.

Finally, from $\Delta z(n) \to \infty$ as $n \to \infty$, we can see that

$$z(n) = z(n_2) + \sum_{s=n_2}^{n-1} \Delta z(s) \leq z(n_2) + (n - n_2)\Delta z(n) \leq n\Delta z(n),$$

which implies

$$\Delta\left[\frac{z(n)}{n}\right] = \frac{n\Delta z(n) - z(n)}{n(n+1)} \geq 0.$$

Thus (iii) holds and hence the proof of the lemmma. □

Lemma 2.5. *Suppose that $y(n)$ is an eventually positive solution of (1) with $z(n)$ satisfying case (I) of Lemma 2.2. Let*

$$\sum_{s=n_0}^{\infty} q(s)\Pi_1^{\alpha}(\sigma(s))h^{\frac{2\alpha}{u}}(s) = \infty. \tag{12}$$

Then

$$\lim_{n\to\infty} \frac{z(n)}{n^{2/u}} = 0. \tag{13}$$

Proof. Since $z(n)$ satisfies case (I) for $n \geq n_2$ for some $n_2 \geq n_1$, by Lemma 2.3, there exists a $n_u \geq n_2$ such that (3) holds for $n \geq n_u$, which implies $z(n)/n^{2/u}$ is decreasing for $n \geq n_u$.
Now we have to claim

$$\lim_{n\to\infty} \frac{z(n)}{n^{2/u}} = 0.$$

If this is not the case, then there exists a constant $b > 0$ and a $n_3 \geq n_u$ such that

$$z(n) \geq bn^{2/u} \text{ for } n \geq n_3. \tag{14}$$

Since case (I) holds, we again arrive at (11) for $n \geq n_3$. Using (14) in (11) gives

$$\Delta^3 z(n) + b^{\alpha} q(n)\Pi_1^{\alpha}(\sigma(n))h^{\frac{2\alpha}{u}}(n) \leq 0 \tag{15}$$

for $n \geq n_4$ for some $n_4 \geq n_3$. Summing (15) from n_4 to $n-1$ gives

$$\sum_{s=n_4}^{n-1} q(s)\Pi_1^{\alpha}(\sigma(s))h^{\frac{2\alpha}{u}}(s) \leq \frac{\Delta^2 z(n_4)}{b^{\alpha}},$$

which contradicts (12) and hence the proof. □

Lemma 2.6. *Assume that $y(n)$ is an eventually positive solution of (1) with $z(n)$ satisfying case (II) of Lemma 2.2. If there exists a nondecreasing sequence $\{\eta(n)\}$ such that $\sigma(n) \leq \eta(n) < \tau(n)$ for $n \geq n_0$ and if*

$$\sum_{s=t_0}^{\infty} q(s)\Pi_2(\sigma(s))(g(s) - h(s))^{2\alpha} = \infty, \tag{16}$$

then

$$\lim_{n\to\infty} \Delta^2 z(n) = 0. \tag{17}$$

Proof. Suppose that $y(n)$ is an eventually positive solution of (1) such that $y(n) > 0, y(\tau(n)) > 0$ and $y(\sigma(n) > 0$ for $n \geq n_1$ for some $n_1 \geq n_0$. As in Lemma 2.4, we again see that (6) and (7) hold. Since $\Delta z(n) < 0$, it follows from (7) that

$$z(\tau^{-1}(n)) \geq z(\tau^{-1}(\tau^{-1}(n))).$$

Thus (6) becomes

$$x(n) \geq \Pi_2(n) z(\tau^{-1}(n)). \tag{18}$$

Using (18) in (1) yields

$$\Delta^3 z(n) + q(n)\Pi_2^{\alpha}(\sigma(n)) z^{\alpha}(h(n)) \leq 0. \tag{19}$$

for $n \geq n_3$ for some $n_3 \geq n_2$.
Since $(-1)^k \Delta^k z(n) > 0$ for $k = 0, 1, 2$ and $\Delta^3 z(n) \leq 0$ for $n_3 \leq r \leq t$, it is seen that

$$z(r) \geq \frac{(t-r)^2}{2} \Delta^2 z(t) \tag{20}$$

Since $\sigma(n) \leq \eta(n)$ and τ is increasing, we conclude that $\tau^{-1}(\sigma(n)) \leq \tau^{-1}(\eta(n))$, i.e, $h(n) \leq g(n)$. Substituting $r = h(n)$ and $t = g(n)$ in (20), we obtain

$$z(h(n)) \geq \frac{(g(n) - h(n))^2}{2} \Delta^2 z(g(n)).$$

Thus (19) becomes,

$$\Delta^3 z(n) + \frac{1}{2^{\alpha}} q(n)\Pi_2^{\alpha}(\sigma(n))(g(n) - h(n))^{2\alpha}(\Delta^2(g(n)))^{\alpha} \leq 0. \tag{21}$$

Since $\Pi_2(n) < 1$, we have $\Pi_2^{\alpha}(n) \geq \Pi_2(n)$. So inequality (21) takes the form

$$\Delta^3 z(n) + \frac{1}{2^{\alpha}} q(n)\Pi_2(\sigma(n))(g(n) - h(n))^{2\alpha}(\Delta^2(g(n)))^{\alpha} \leq 0. \tag{22}$$

Now we claim that (16) implies $\Delta^2 z(n) \to 0$ as $n \to \infty$. Suppose to the contrary that $\lim\limits_{n\to\infty} \Delta^2 z(n) = l > 0$. Then $\Delta^2 z(n) \geq l$ for $n \geq n_3$ for some $n_3 \geq n_2$. Since $\lim\limits_{n\to\infty} g(n) = \infty$, we can choose $n_4 \geq n_3$ such that $g(n) \geq n_3$ for all $n \geq n_4$. Hence $\Delta^2 g(n) \geq l$ for $n \geq n_4$. Using this in (22) gives

$$\Delta^3 z(n) + \frac{l^{\alpha}}{2^{\alpha}} q(n)\Pi_2(\sigma(n))(g(n) - h(n))^{2\alpha} \leq 0 \tag{23}$$

for $n \geq n_4$. Summing (23) from n_4 to $n-1$ gives

$$\sum_{s=n_4}^{n-1} q(s)\Pi_2(\sigma(s))(g(s) - h(s))^{2\alpha} \leq \left(\frac{2}{l}\right)^{\alpha} \Delta^2 z(n_4)$$

which is a contradiction to (16). This completes the proof. □

Now the following theorem is concerned with equation (1) when $\alpha = 1$.

Theorem 2.7. *Let (4) hold and suppose that there exists a nondecreasing sequence* $\{\eta(n)\}$ *such that* $\sigma(n) \leq \eta(n) < \tau(n)$ *for* $n \geq n_0$. *If there exist constants* v, u *such that* $0 < v, u \leq 1$ *satisfying*

$$\limsup_{n\to\infty}\left(\frac{vuh^{1-\frac{2}{u}}(n)}{2}\sum_{s=n_0}^{h(n)} sq(s)\Pi_1(\sigma(s))(h(s))^{\frac{2}{u}}\right) + \limsup_{n\to\infty}\left(\frac{vuh^{2-\frac{2}{u}}(n)}{2}\sum_{s=h(n)}^{n-1} q(s)\Pi_1(\sigma(s))(h(s))^{\frac{2}{u}}\right) + \limsup_{n\to\infty}\left(\frac{vuh(n)}{2}\sum_{s=n-1}^{\infty} q(s)\Pi_1(\sigma(s))h(s)\right) > 1 \tag{24}$$

and

$$\limsup_{n\to\infty}\sum_{s=g(n)}^{n-1}\frac{1}{2}q(s)\Pi_2(\sigma(s))(g(s)-h(s))^2 > 1 \tag{25}$$

then all the solutions of equation (1) are oscillatory.

Proof. Suppose that $y(n)$ is a nonoscillatory solution of (1), say $y(n) > 0, y(\tau(n)) > 0$ and $y(\sigma(n) > 0$ for $n \geq n_1$ for some $n_1 \geq n_0$. Then from Lemma 2.2, the corresponding sequence z satisfies either case (I) or case (II) for $n \geq n_2$ for some $n_2 \geq n_1$.

First we consider case(I). By Lemma 2.4, we again arrive at (11) for $n \geq n_3$ which, for $\alpha = 1$, takes the form

$$\Delta^3 z(n) + q(n)\Pi_1(\sigma(n))z(h(n)) \leq 0 \text{ for } n \geq n_3. \tag{26}$$

Summing (26) from n to ∞ gives

$$\Delta^2 z(n) \geq \sum_{s=n}^{\infty} q(s)\Pi_1(\sigma(s))z(h(s)), \tag{27}$$

and summing again from n_3 to $n-1$ gives

$$\begin{aligned}\Delta z(n) &\geq \sum_{u=n_3}^{n-1}\sum_{s=u}^{\infty} q(s)\Pi_1(\sigma(s))z(h(s)) \\ &= \sum_{u=n_3}^{n-1}\sum_{s=u}^{n-1} q(s)\Pi_1(\sigma(s))z(h(s)) + \sum_{u=n_3}^{n-1}\sum_{s=n-1}^{\infty} q(s)\Pi_1(\sigma(s))z(h(s)) \\ &= \sum_{s=n_3}^{n-1}(s-n_3)q(s)\Pi_1(\sigma(s))z(h(s)) + (n-n_3)\sum_{s=n-1}^{\infty} q(s)\Pi_1(\sigma(s))z(h(s)).\end{aligned}$$

For any $0 < v \leq 1$, there exists $n_4 \geq n_3$ such that $s - n_3 \geq vs$ and $n - n_3 \geq vn$ for $n \geq s \geq n_4$. Thus from the last inequality, we obtain

$$\Delta z(n) \geq \alpha \sum_{s=n_4}^{n-1} sq(s)\Pi_1(\sigma(s))z(h(s)) + \alpha n \sum_{s=n-1}^{\infty} q(s)\Pi_1(\sigma(s))z(h(s)). \tag{28}$$

Using (3*) in (28) gives

$$\frac{2z(n)}{un} \geq \alpha \sum_{s=n_4}^{n-1} sq(s)\Pi_1(\sigma(s))z(h(s)) + \alpha n \sum_{s=n-1}^{\infty} q(s)\Pi_1(\sigma(s))z(h(s)). \tag{29}$$

From (29), we obtain

$$\begin{aligned}\frac{2z(h(n))}{uh(n)} \geq \alpha &\sum_{s=n_4}^{h(n)} sq(s)\Pi_1(\sigma(s))z(h(s)) \\ &+ \alpha h(n) \sum_{s=h(n)}^{n-1} q(s)\Pi_1(\sigma(s))z(h(s)) \\ &+ \alpha h(n) \sum_{s=n-1}^{\infty} q(s)\Pi_1(\sigma(s))z(h(s)).\end{aligned} \tag{30}$$

Also for $n \leq s$, we have $h(n) \leq h(s)$. Since $z(n)/n$ is increasing,

$$z(h(s)) \geq \frac{h(s)z(h(n)}{h(n)}. \tag{31}$$

For $h(n) \leq s \leq n$, we have $h(h(n)) \leq h(s) \leq h(n)$. Since $\frac{z(n)}{n^{2/u}}$ is decreasing,

$$z(h(s)) \geq \frac{h^{2/u}(s)z(h(n)}{h^{2/u}(n)}. \tag{32}$$

For $n_4 \leq s \leq h(n)$ and $h(n) < n$, we have $h(s) \leq h(h(n) < h(n)$. Since $\frac{z(n)}{n^{2/u}}$ is decreasing, we again obtain (32). Using (31) and (32) in (30) gives

$$\begin{aligned}\frac{2z(h(n))}{uh(n)} \geq &\left(\alpha \sum_{s=n_4}^{h(n)} sq(s)\Pi_1(\sigma(s))(h(s))^{2/u}\right) \frac{zh(n)}{(h(n))^{2/u}} \\ &+ \left(\alpha h(n) \sum_{s=h(n)}^{n-1} q(s)\Pi_1(\sigma(s))(h(s))^{2/u}\right) \frac{zh(n)}{(h(n))^{2/u}} \\ &+ \left(\alpha h(n) \sum_{s=n-1}^{\infty} q(s)\Pi_1(\sigma(s))h(s)\right) \frac{z(h(s)}{h(s)}.\end{aligned} \tag{33}$$

From (33), we see that

$$\begin{aligned}
&\frac{vuh^{1-\frac{2}{u}}(n)}{2}\sum_{s=n_4}^{h(n)} sq(s)\Pi_1(\sigma(s))(h(s))^{\frac{2}{u}}\\
&\qquad+\frac{vuh^{2-\frac{2}{u}}(n)}{2}\sum_{s=h(n)}^{n-1} q(s)\Pi_1(\sigma(s))(h(s))^{\frac{2}{u}}\\
&\qquad+\frac{vuh(n)}{2}\sum_{s=n-1}^{\infty} q(s)\Pi_1(\sigma(s))h(s)\leq 1.
\end{aligned}$$

Taking limit supremum on both sides of above inequality, we obtain a contradiction to (24).

Next we consider case (II). Proceeding as in Lemma 2.6, we again arrive at (21), which for $\alpha = 1$ becomes

$$\Delta^3 z(n)+\frac{1}{2}q(n)\Pi_2(\sigma(n))(g(n)-h(n))^2\Delta^2(g(n))\leq 0. \tag{34}$$

Summing (34) from $g(n)$ to $n-1$ gives

$$\Delta^3 z(n)+\left[\sum_{s=g(n)}^{n-1}\frac{1}{2}q(s)\Pi_2(\sigma(s))(g(s)-h(s))^2\right]\Delta^2(g(n))\leq 0,$$

which is a contradiction to (25). This completes the proof. □

Next theorem provides the oscillatory results for equation (1) in the case when $\alpha < 1$.

Theorem 2.8. *Assume that (4) and (12) hold. Suppose there exists a nondecreasing sequence $\eta(n)$ such that $\sigma(n) \leq \eta(n) \leq \tau(n)$ for $n \geq n_0$. If there exists $0 < u \leq 1$ such that limsup of*

$$\begin{aligned}
&h^{1-\frac{2}{u}}(n)\sum_{s=n_0}^{h(n)} sq(s)\Pi_1^{\alpha}(\sigma(s))(h(s))^{\frac{2\alpha}{u}}\\
&\quad+h^{2-\frac{2}{u}}(n)\sum_{s=h(n)}^{n-1} q(s)\Pi_1^{\alpha}(\sigma(s))(h(s))^{\frac{2\alpha}{u}}\\
&\quad+\frac{h^{2-\alpha}(n)}{h^{\frac{2(1-\alpha)}{u}}(n)}\sum_{s=n-1}^{\infty} q(s)\Pi_1^{\alpha}(\sigma(s))h^{\alpha}(s)
\end{aligned}\tag{35}$$

and

$$\sum_{s=g(n)}^{n-1} q(s)\Pi_2(\sigma(s))(g(s)-h(s))^{2\alpha} \tag{36}$$

are greater than zero as $n\to\infty$, then all the solutions of equation (1) are oscillatory.

Proof. Suppose that $y(n)$ is a nonoscillatory solution of (1), say $y(n) > 0, y(\tau(n)) > 0$ and $y(\sigma(n) > 0$ for $n \geq n_1$ for some $n_1 \geq n_0$. Then from Lemma 2.2, the corresponding sequence $z(n)$ satisfies either case (I) or case (II) for $n \geq n_2$ for some $n_2 \geq n_1$.

First we consider case (I). By Lemma 2.4, we again arrive at (11) for $n \geq n_3$. Summing (11) from n to ∞ gives

$$\Delta^2 z(n) \geq \sum_{s=n}^{\infty} q(s)\Pi_1^\alpha(\sigma(s))z^\alpha(h(s)) \text{ for } n \geq n_3. \tag{37}$$

Summing (37) from n_3 to $n-1$ gives

$$\begin{aligned}\Delta z(n) &\geq \sum_{u=n_3}^{n-1}\sum_{s=u}^{\infty} q(s)\Pi_1^\alpha(\sigma(s))z^\alpha(h(s)) \\ &= \sum_{u=n_3}^{n-1}\sum_{s=u}^{n-1} q(s)\Pi_1^\alpha(\sigma(s))z^\alpha(h(s)) + \sum_{u=n_3}^{n-1}\sum_{s=n-1}^{\infty} q(s)\Pi_1^\alpha(\sigma(s))z^\alpha(h(s)) \\ &= \sum_{s=n_3}^{n-1}(s-n_3)q(s)\Pi_1^\alpha(\sigma(s))z^\alpha(h(s)) + (n-n_3)\sum_{s=n-1}^{\infty} q(s)\Pi_1^\alpha(\sigma(s))z^\alpha(h(s)).\end{aligned}$$

For any $0 < v \leq 1$, there exists $n_4 \geq n_3$ such that $s - n_3 \geq vs$ and $n - n_3 \geq vn$ for $n \geq s \geq n_4$. Thus we obtain

$$\Delta z(n) \geq \alpha \sum_{s=n_4}^{n-1} sq(s)\Pi_1^\alpha(\sigma(s))z^\alpha(h(s)) + \alpha n \sum_{s=n-1}^{\infty} q(s)\Pi_1^\alpha(\sigma(s))z^\alpha(h(s)). \tag{38}$$

Using (3*) in (38) gives

$$\frac{2z(n)}{un} \geq \alpha \sum_{s=n_4}^{n-1} sq(s)\Pi_1^\alpha(\sigma(s))z^\alpha(h(s)) + \alpha n \sum_{s=n-1}^{\infty} q(s)\Pi_1^\alpha(\sigma(s))z^\alpha(h(s)). \tag{39}$$

From (39), we obtain

$$\begin{aligned}\frac{2z(h(n))}{uh(n)} \geq \alpha &\sum_{s=n_4}^{h(n)} sq(s)\Pi_1^\alpha(\sigma(s))z^\alpha(h(s)) \\ &+ \alpha h(n) \sum_{s=h(n)}^{n-1} q(s)\Pi_1^\alpha(\sigma(s))z^\alpha(h(s)) \\ &+ \alpha h(n) \sum_{s=n-1}^{\infty} q(s)\Pi_1^\alpha(\sigma(s))z^\alpha(h(s)).\end{aligned} \tag{40}$$

Using (31) and (32) in (40) yields

$$\begin{aligned}\frac{2z(h(n))}{uh(n)} \geq & \left(\alpha \sum_{s=n_4}^{h(n)} sq(s)\Pi_1^\alpha(\sigma(s))(h(s))^{2\alpha/u}\right) \frac{z^\alpha h(n)}{h^{2\alpha/u}(n)} \\ & + \left(\alpha h(n) \sum_{s=h(n)}^{n-1} q(s)\Pi_1^\alpha(\sigma(s))(h(s))^{2\alpha/u}\right) \frac{z^\alpha h(n)}{h^{2\alpha/u}(n)} \\ & + \left(\alpha h(n) \sum_{s=n-1}^{\infty} q(s)\Pi_1^\alpha(\sigma(s))h^\alpha(s)\right) \frac{z^\alpha(h(n)}{h^\alpha(n)}. \end{aligned} \tag{41}$$

Let $w(n) = \frac{z(h(n))}{h^{2/\alpha}(n)}$. Then from (41), we obtain

$$\begin{aligned}\frac{2}{\alpha u} w^{1-\alpha}(n) \geq & h^{1-\frac{2}{u}}(n) \left(\sum_{s=n_4}^{h(n)} sq(s)\Pi_1^\alpha(\sigma(s))(h(s))^{2\alpha/u}\right) \\ & + h^{2-\frac{2}{u}}(n) \left(\sum_{s=h(n)}^{n-1} q(s)\Pi_1^\alpha(\sigma(s))(h(s))^{2\alpha/u}\right) \\ & + \frac{h^{2-\alpha}(n)}{h^{\frac{2(1-\alpha)}{u}}(n)} \left(\sum_{s=n-1}^{\infty} q(s)\Pi_1^\alpha(\sigma(s))h^\alpha(s)\right). \end{aligned} \tag{42}$$

Taking limsup as $n \to \infty$ on both sides of the above inequality and using (13), we obtain a contradiction to (35).

Next, we consider case (II). Proceeding as in the proof of Lemma 2.6, we again arrive at (22). Summing (22) from $g(n)$ to $n-1$ gives

$$\sum_{s=g(n)}^{n-1} q(s)\Pi_2(\sigma(s))(g(s) - h(s))^{2\alpha} \leq 2^\alpha(\Delta^2(g(n)))^{1-\alpha}$$

Noting that (36) implies (16), we see that (17) holds. Taking the limsup as $n \to \infty$ on both sides of the above inequality and using (17), we obtain a contradiction to (36) and this completes the proof of the theorem. □

The following are the examples which illustrate the main results.

3 Examples

First example establishes the equation with bounded neutral coefficients.

Example 3.1. *Consider the third order difference equation*

$$\Delta^3\left[y(n) + 32y(\frac{n}{2})\right] + \frac{1}{n^3}y(\frac{n}{4}) = 0, n \geq 1 \tag{E1}$$

Here $p(n) = 32$, $q(n) = \frac{1}{n^3}$, $\alpha = 1$, $\tau(n) = \frac{n}{2} < n$ *and* $\sigma(n) = \frac{n}{4}$.

Then we can see that conditions (H1)-(H2) hold and

$\tau^{-1}(n) = 2n, \tau^{-1}(\tau^{-1}((n)) = 4n, h(n) = \frac{n}{2}$ *and* $g(n) = \frac{2n}{3}$ *with* $\eta(n) = \frac{n}{3}$.

Set $u = 2/3$. *Then we get*

$$\left(\frac{n}{\tau(n)}\right)^{2/u} \frac{1}{p(n)} = \frac{1}{2}.$$

Thus condition (H3) holds, $\Pi_1(n) = 1/64$ *and* $\Pi_2(n) = 31/1024$.

Letting $v = u = 2/3$, *we can easily see that all conditions of Theorem 2.7 are satisfied and hence all the solutions of equation (E1) are oscillatory.*

The second example is concerned with an equation with unbounded neutral coefficients.

Example 3.2. *Consider the sublinear difference equation*

$$\Delta^3 \left[y(n) + 2ny(\frac{n}{2})\right] + \frac{1}{n^{6/5}} y^{3/5}(\frac{n}{10}) = 0, n \geq 8. \tag{E2}$$

Here $p(n) = 2n$, $q(n) = \frac{1}{n^{6/5}}$, $\alpha = 3/5$, $\tau(n) = n/2 < n$ *and* $\sigma(n) = n/10$.

Then conditions (H1)-(H2) hold and $\tau^{-1}(n) = 2n, \tau^{-1}(\tau^{-1}((n)) = 4n$,

$h(n) = \tau^{-1}(\sigma(n)) = \frac{n}{5}$ *and* $g(n) = \tau^{-1}(\eta(n)) = \frac{n}{4}$ *with* $\eta(n) = \frac{n}{8}$.

Choosing $u = 2/3$, *we get*

$$\left(\frac{n}{\tau(n)}\right)^{2/u} \frac{1}{p(n)} = \frac{4}{n} \leq \frac{1}{2},$$

i.e., condition (H3) holds. Since $\Pi_1(n) \geq \frac{7}{32n}$ *and* $\Pi_2(n) \geq \frac{63}{256n}$,

by Theorem 2.8, equation (E2) is oscillatory.

4 Conclusion

In this paper, by using the summing averaging technique, the oscillatory behaviour of every solution of the equation (1) are discussed in Theorems 2.7 and 2.8. Here some sufficient conditions for all solutions to be oscillatory are proved. These sufficient conditions which are new, extend and complement some of the known results in the literature. Also the examples reveal the illustration of the proved results.

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Biography

S.Kaleeswari was born in Pollachi, Tamil Nadu, India. She received her B.Sc., degree in Mathematics from NGM College, Bharathiar University, Coimbatore in 1999, her M.Sc., degree in Mathematics from NGM College, Bharathiar University, Coimbatore in 2001, her M.Phil., degree in Mathematics from Bharathiar University, Coimbatore in 2003 and her Ph.D degree in Mathematics from Anna University, Chennai in 2017. She is doing her research in the field of difference and differential equations.

She is an Assistant Professor in the Department of Mathematics, Nallamuthu Gounder Mahalingam College, Pollachi. She has 16 years of experience in teaching and She has published 15 papers in national and international journals and one book chapter. She published her results on oscillation theory of ordinary and delay difference equations. She has life membership in Indian mathematical society, Ramanujan mathematical society, Indian science congress association. She acts as reviewer for some refereed journals.

Said. R. Grace was born in Port-Said, Egypt. He received his B.S., degree in Electrical from Cairo University, Orman, Egypt in 1970, his B.S., degree in Mathematics from Cairo University, Orman, Egypt in 1973, his M.S., degree in Mathematics from University of
Saskatchewan, Saskatoon, Canada in 1978 and his Ph.D degree in Mathematics from University of Saskatchewan, Saskatoon, Canada in 1981. His general areas of research are ordinary and functional differential equations, difference equations, impulsive systems, differential inclusions, integral equations, dynamic equations on time scales, and their applications and fractional differential equations.

At present he is a Professor of Mathematics in the Department of Engineering Mathematics, Faculty of Engineering, Cairo University, Orman, Giza 12221, Egypt. He published more than 400 papers in refereed journals and some books. A selection of some more-recent publications indicating various research interests can be found in Math Sci Net. His H-Index is 26 of the 255 documents. Some of his books are listed below: 1. Oscillation theory for difference and functional differential equations (Kluwer Academic Publishers, Dordrecht, 2000. viii+337 pp. ISBN: 0-7923-6289- 6). 2. Oscillation theory for second order dynamic equations. Series in Mathematical Analysis and Applications, 5 (Taylor and Francis, Ltd., London, 2003. viii+404 pp. ISBN: 0-415-30074-6). 3. Discrete oscillation theory (Hindawi Publishing Corporation, New York, 2005. xiv+961 pp. ISBN: 977-5945-19-4).