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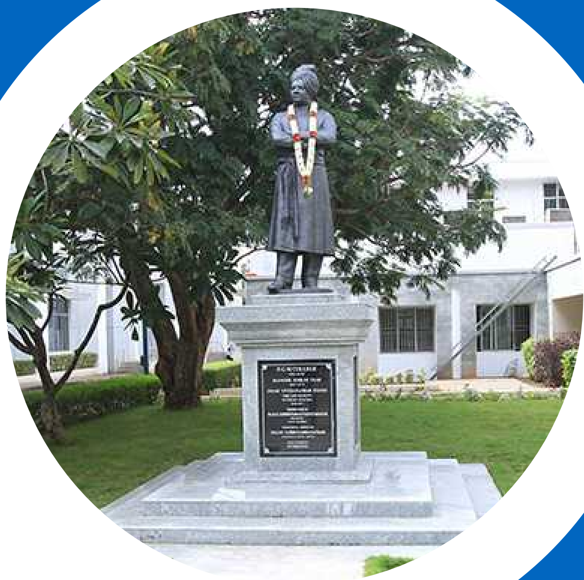

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EMERGING TRENDS IN SCIENCE AND TECHNOLOGY (ETIST-2021)

27th October 2021

Jointly Organized by

Department of Biological Science, Physical Science and Computational Science

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Nallamuthu Gounder Mahalingam College, Pollachi- 1

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ABOUT THE INSTITUTION

A nations's growth is in proportion to education and intelligence spread among the masses. Having this idealistic vision, two great philanthropists late. S.P. Nallamuthu Gounder and Late. Arutchelver Padmabhushan Dr.N.Mahalingam formed an organization called Pollachi Kalvi Kazhagam, which started NGM College in 1957, to impart holistic education with an objective to cater to the higher educational needs of those who wish to aspire for excellence in knowledge and values. The College has achieved greater academic distinctions with the introduction of autonomous system from the academic year 1987-88. The college has been Re-Accredited by NAAC and it is ISO 9001 : 2015 Certified Institution. The total student strength is around 6000. Having celebrated its Diamond Jubilee in 2017, the college has blossomed into a premier Post-Graduate and Research Institution, offering 26 UG, 12 PG, 13 M.Phil and 10 Ph.D Programmes, apart from Diploma and Certificate Courses. The college has been ranked within Top 100 (72nd Rank) in India by NIRF 2021.

ABOUT CONFERENCE

The International conference on “Emerging Trends in Science and Technology (ETIST-2021)” is being jointly organized by Departments of Biological Science, Physical Science and Computational Science - Nallamuthu Gounder Mahalingam College, Pollachi along with ISTE, CSI, IETE, IEE & RIYASA LABS on 27th OCT 2021. The Conference will provide common platform for faculties, research scholars, industrialists to exchange and discus the innovative ideas and will promote to work in interdisciplinary mode.

Oscillatory Behavior of Nonlinear Fourth Order Mixed Neutral Difference Equations

S. Kaleeswari[1] - M. Buvanasankari[2]

ABSTRACT

The objective of this paper is to study the oscillatory criteria for nonlinear fourth order difference equation with mixed neutral terms of the form

$$\Delta(q_1(n)(\Delta^3 z(n))^{\alpha_1}) = q_2(n)y^{\alpha_2}(n-m+1) + q_3(n)y^{\alpha_3}(n+m^*)$$

where z(n) = y(n) + $q_4(n)\,y^{\alpha_4}(n-k) + q_5(n)y^{\alpha_5}(n-k)$.

Here $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ are the ratios of odd positive integers $\alpha_1 \geq 1$, q_1, q_2, q_3, q_4, q_5 are positive sequences and $m, m^*, k \in \mathbb{N}$ are such that m > 3, $m^* > 3, k < m-2$. By means of comparison techniques, we obtain some new oscillation results. Examples are given to illustrate the important of the results.

Keywords: comparison techniques, difference equations, fourth order, mixed neutral terms, nonlinear, oscillation.

2020 Mathematics Subject Classification: 39A10

1. INTRODUCTION

The problem of determining oscillation criteria for mixed neutral difference equations has been receiving great attention in the last few decades, since these types of equations arise in the study of economics, mathematical biology, and many other areas of mathematics [1, 2, 11, 12, 17]. Some interesting recent results on the oscillatory behavior of second-order difference equations can be found in [4, 20, 22, 23, 26] and the references cited therein.

From a review of literature, it is found that all the results established for fourth order difference equations with mixed neutral terms are guaranteed that every solution is either oscillatory or tends to zero monotonically, and to the best of our knowledge, there are no results in the literature which ensure that all solutions are just oscillatory for the fourth order mixed neutral difference equations.

Therefore, the purpose of this paper is to present some new oscillation criteria for equation of the form

$$\Delta\left(q_1(n)(\Delta^3 z(n))^{\alpha_1}\right) = q_2(n)y^{\alpha_2}(n-m+1) + q_3(n)y^{\alpha_3}(n+m^*) \tag{1}$$

where $z(n) = y(n) + q_4(n)y^{\alpha_4}(n-k) + q_5(n)y^{\alpha_5}(n-k)$.

via comparison with the first-order equations whose oscillatory behavior are known, or via comparison with second-order difference equations with neutral terms. For relevant results on the applications of oscillation theory, the reader can refer [5, 6, 7]

The following conditions are always assumed to hold:

(i) $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ are the ratios of odd positive integers $\alpha_1 \geq 1$.

(ii) $q_1, q_2, q_3, q_4, q_5 : \mathbb{Z} \to (0, \infty)$ are sequences.

Dr. S. Kaleeswari[1], Department of Mathematics, Nallamuthu Gounder Mahalingam College, Pollachi, Coimbatore, Tamilnadu, India. kaleesdesika@gmail.com
M. Buvanasankari[2], Department of Mathematics, Nehru Institute of Engineering and Technology, Coimbatore, Tamilnadu, India. buvanasankari@gmail.com

(iii) m, m^*,k $\in \mathbb{N}$ are such that m > 3, m^*> 3 , k < m – 2.

A solution of "Equation (1)" is called oscillatory if it is neither eventually negative nor eventually positive. Otherwise it is said to be non-oscillatory. If all the solutions of the "Equation (1)" are oscillatory, then the equation itself is oscillatory.

The objective of the this paper is to provide sufficient conditions for the oscillation of "Equation (1)" whenever $\alpha_5 > 1 > \alpha_4$ or $1 \geq \alpha_5 > \alpha_4$ and subject to the assumption

$$Q_1(n,n_1) \to \infty \ as\ n \to \infty, \text{ where } Q_1(t,s) = \sum_{r=s}^{t-1} \frac{1}{q_1^{\frac{1}{\alpha_1}}(r)}. \tag{2}$$

2. AUXILIARY RESULTS

Lemma 2.1 [see 7, Lemma 1 & 19, Lemma 2.2]

(I) If the first order delay difference inequality

$$\Delta z(n) + q_2(n) f(z(n-m+1)) \leq 0$$

has an eventually positive solution, then so does the corresponding delay difference equation.

(II) If the first order advanced difference inequality

$$\Delta z(n) - q_2(n) f(z(n-m^*)) \geq 0$$

has an eventually positive solution, then so does the corresponding advanced difference equation.

Lemma 2.2 (see [9]) If X,Y ≥ 0, then

$$X^\lambda + (\lambda-1)Y^\lambda - \lambda X Y^{\lambda-1} \geq 0, \qquad for\ \ \lambda > 1. \tag{3}$$

and

$$X^\lambda + (1-\lambda)Y^\lambda - \lambda X Y^{\lambda-1} \leq 0, \qquad for\ \ 0 < \lambda < 1. \tag{4}$$

Lemma 2.3

Assume "Equation (2)". Then $\Delta Z(n) > 0$ eventually, where Z = $q_1(\Delta^3 z)^{\alpha_1}$ (5)

implies that one of the following four cases happen:

Case (I) : $z(n) > 0$, $\Delta z(n) > 0$, $\Delta^2 z(n) > 0$, $\Delta^3 z(n) > 0$

Case (II) : $z(n) > 0$, $\Delta z(n) > 0$, $\Delta^2 z(n) > 0$, $\Delta^3 z(n) < 0$

Case (III) : $z(n) < 0$, $\Delta z(n) < 0$, $\Delta^2 z(n) < 0$, $\Delta^3 z(n) < 0$

Case (IV) : $z(n) < 0$, $\Delta z(n) > 0$, $\Delta^2 z(n) < 0$, $\Delta^3 z(n) < 0$

Proof:

From "Equation (5)", we can find $n_0 \in \mathbb{N}_0$ such that $\Delta Z(n) > 0$ for all n $\geq n_0$. (6)

We suppose that, there exists $n_1 \geq n_0$ with $Z(n_1) > 0$. (7)

From "Equation (6)" and "Equation (7)", we get, for all $n \geq n_1$,

$$Z(n) = Z(n_1) + \textstyle\sum_{r=n_1}^{n-1} \Delta Z(r) \geq Z(n_1) > 0.$$

Thus, $\Delta^3 z(n) > 0$ for all $n \geq n_1$. (8)

From "Equation (6)" and "Equation (7)", we obtain, for $n \geq n_1$,

$$\Delta^2 z(n) = \Delta^2 z(n_1) + \sum_{r=n_1}^{n-1} \Delta^3 z(r) = \Delta^2 z(n_1) + \sum_{r=n_1}^{n-1} \frac{Z^{\frac{1}{\alpha_1}}(r)}{q_1^{\frac{1}{\alpha_1}}(r)}$$

$$\geq \Delta^2 z(n_1) + \sum_{r=n_1}^{n-1} \frac{z^{\frac{1}{\alpha_1}}(n_1)}{q_1^{\frac{1}{\alpha_1}}(n_1)}$$

$$= \Delta^2 z(n_1) + Z^{\frac{1}{\alpha_1}}(n_1) Q_1(n, n_1) \longrightarrow \infty \ as \ n \to \infty.$$

because of “Equation (2)”.

Therefore, there exists $n_2 \geq n_1$ with $\Delta^2 z(n) > 0$ for all $n \geq n_2$. (9)

From “Equation (8)” and “Equation (9)”, we get, for $n \geq n_2$,

$$\Delta z(n) = \Delta z(n_2) + \sum_{r=n_2}^{n-1} \Delta^2 z(r) \geq \Delta z(n_2) + \sum_{r=n_2}^{n-1} \Delta^2 z(n_2)$$

$$= \Delta z(n_2) + (n - n_2)\Delta^2 z(n_2) \to \infty \quad as \quad n \to \infty.$$

Hence, there exists $n_3 \geq n_2$ with $\Delta z(n) > 0$ for all $n \geq n_3$. (10)

From “Equation (9)” and “Equation (10)”, we obtain, for $n \geq n_3$,

$$z(n) = z(n_3) + \sum_{r=n_3}^{n-1} \Delta z(r) \geq z(n_3) + \sum_{r=n_3}^{n-1} \Delta z(n_3)$$

$$= z(n_3) + (n - n_3)\Delta z(n_3) \to \infty \quad as \quad n \to \infty.$$

Hence, there exists $n_4 \geq n_3$ with $z(n) > 0$ for all $n \geq n_4$. (11)

By “Equations (8-11)”, we get

$$z(n) > 0, \ \Delta z(n) > 0, \ \Delta^2 z(n) > 0, \ \Delta^3 z(n) > 0 \ \text{for all } n \geq n_4.$$

Thus, Case (I) holds. If “Equation (7)” does not hold, then the only other possibility is

$Z(n) < 0$ for all n $\geq n_0$.

and thus, $\Delta^3 z(n) < 0$ for all n $\geq n_0$. (12)

We suppose that, there exists $n_1 \geq n_0$ with $\Delta^2 z(n_1) < 0$. (13)

From “Equation (12)” and “Equation (13)”, we get, for all $n \geq n_1$,

$$\Delta^2 z(n) = \Delta^2 z(n_1) + \sum_{r=n_1}^{n-1} \Delta^3 z(r) = \Delta^2 z(n_1) + \sum_{r=n_1}^{n-1} \frac{Z^{\frac{1}{\alpha_1}}(r)}{q_1^{\frac{1}{\alpha_1}}(r)}$$

$$\leq \Delta^2 z(n_1) < 0.$$

Hence, $\Delta^2 z(n) < 0$ for all $n \geq n_1$. (14)

Now, from “Equation (12)” and “Equation (13)”, we obtain, for $n \geq n_1$,

$$\Delta z(n) = \Delta z(n_1) + \sum_{r=n_1}^{n-1} \Delta^2 z(r) \leq \Delta z(n_1) + \sum_{r=n_1}^{n-1} \Delta^2 z(n_1)$$

$$\leq \Delta z(n_1) + (n - n_1)\Delta^2 z(n_1) \to \infty \ as \ n \to \infty.$$

Hence, there exists $n_2 \geq n_1$ with $\Delta z(n) < 0$ for all $n \geq n_2$. (15)

Now, from “Equation (14)” and “Equation (15)”, we obtain, for $n \geq n_2$,

$$z(n) = z(n_2) + \sum_{r=n_2}^{n-1} \Delta z(r) \leq z(n_2) + \sum_{r=n_2}^{n-1} \Delta z(n_2)$$

$$\leq z(n_2) + (n - n_2)\Delta z(n_2) \to \infty \quad as \quad n \to \infty.$$

Thus, there exists $n_3 \geq n_2$ with $z(n) < 0$ for all $n \geq n_3$. (16)

By “Equations (12-16)”, we get

$$z(n) < 0, \ \Delta z(n) < 0, \ \Delta^2 z(n) < 0, \ \Delta^3 z(n) < 0.$$

Thus, Case (III) holds. Further, if “Equation (13)” does not hold, then the only other possibility is

$\Delta^2 z(n_1) > 0$ for all n $\geq n_0$. (17)

Suppose that, there exists $n_1 \geq n_0$ with $\Delta z(n_1) > 0$. (18)

Then, from "Equation (17)" and "Equation (18)", we get, for all $n_1 \geq n_0$,

$$\Delta z(n) = \Delta z(n_1) + \sum_{r=n_1}^{n-1} \Delta^2 z(r) \geq \Delta z(n_1) > 0.$$

Therefore, there exists $n_2 \geq n_1$ with $\Delta z(n) > 0$ for all $n \geq n_2$. (19)

Now, from "Equation (18)" and "Equation (19)", for $n \geq n_2$,

$$z(n) = z(n_2) + \sum_{r=n_2}^{n-1} \Delta z(r) \geq z(n_2) + \sum_{r=n_2}^{n-1} \Delta z(n_2)$$

$$\geq z(n_2) + (n - n_2)\Delta z(n_2) \to \infty \quad as \quad n \to \infty.$$

Hence, there exists $n_3 \geq n_2$ with $z(n) > 0$ for all $n \geq n_3$. (20)

By "Equation (12)", "Equation (17)", "Equation (19)" and "Equation (20)", we get

$$z(n) > 0,\ \Delta z(n) > 0,\ \Delta^2 z(n) > 0,\ \Delta^3 z(n) < 0.$$

Thus, Case (II) holds. Finally, if "Equation (15)" does not hold, then the only possibility is

$\Delta z(n) > 0$ for all $n \geq n_0$. (21)

By "Equation (12)", "Equation (14)", "Equation (16)" and "Equation (21)", we get

$$z(n) < 0,\ \Delta z(n) > 0,\ \Delta^2 z(n) < 0,\ \Delta^3 z(n) < 0.$$

Thus, Case (IV) holds.

In the rest of the paper, we assume that $k_0, k_1, k_2, k_3, k_4 \in \mathbb{N}$ satisfy $3k_0 < m^*$, $k_1 < m + 2$

and $k_2 < k_3 < k_4 \leq m+1-k$ (22)

Note 2.4:

1. Consider the assumptions $m > 3$, $m^* > 3$ and $k < m-2$. e.g., one may choose $k_0 = k_1 = k_2 = 1$, $k_3 = 2$, and $k_4 = 4$.
2. Consider that $n + m^* - 3k_0 > n$ holds always since $m^* - 3k_0 > 0$. Therefore, equations involving $n + m^* - 3k_0$ are of advanced type. Furthermore, $n-m+k_1-2 < n$, $n-m+k-2 < n$, $n-m+k-2+k_3 < n$, always since $m+2-k_1 > 0$, $m+2-k > 0$, $m+2-k-k_3 > 0$. Therefore, equation containing $n-m+k_1-2$, $n-m+k-2$, $n-m+k-2+k_3$ are of delay type.

3 MAIN RESULTS

We begin with following new result.

Theorem 3.1

Let $\alpha_5 > 1 > \alpha_4$. Assume that (i) – (iii), "Equation (2)" and "Equation (22)" hold. Suppose that there exists $q: \mathbb{Z} \to (0, \infty)$, such that $\lim_{n\to\infty} (g_1(n) + g_2(n)) = 0$ where

$$g_1(n) = (1 - \alpha_4)\alpha_4^{\frac{\alpha_4}{1-\alpha_4}} q^{\frac{\alpha_4}{\alpha_4 - 1}}(n) q_4^{\frac{1}{1-\alpha_4}}(n) \quad \text{and}$$

$$g_2(n) = (\alpha_5 - 1)\alpha_5^{\frac{\alpha_5}{1-\alpha_5}} q^{\frac{\alpha_5}{\alpha_5 - 1}}(n) q_5^{\frac{1}{1-\alpha_5}}(n) \tag{23}$$

Let $\theta_0, \theta_1, \theta_2 \in (0,1)$. If the first order advanced difference equation

$$\Delta z(n) = \theta_0 z^{\frac{\alpha_3}{\alpha_1}}(n + m^* - 3k_0) \sum_{r=n-k_0}^{n-1} \left(\sum_{t=r-k_0}^{r-1} \left(\frac{1}{q_1(t)} \sum_{\xi=t-k_0}^{t-1} q_3(\xi) \right)^{\frac{1}{\alpha_1}} \right), \tag{24}$$

and the first order delay difference equations

$$\Delta X(n)+(\theta_1\theta_2 k)^{\alpha_2} q_2(n)(n-m+1)^{\alpha_2}(n-m)^{\alpha_2} X^{\frac{\alpha_2}{\alpha_1}}(n-m+k_1-2) \qquad (25)$$
$$[Q_1(n-m+k_1-1,n-m-1)]^{\alpha_2}=0,$$

$$\Delta X(n)+\frac{q_2(n)}{q_5^{\frac{\alpha_2}{\alpha_5}}(n-m+k+1)} X^{\frac{\alpha_2}{\alpha_1\alpha_5}}(n-m+k-2)[\sum_{r=n_1}^{n-m+k}(\sum_{t=r_1}^{r-m+k}Q_1(t,n_1))]^{\frac{\alpha_2}{\alpha_5}}=0, \qquad (26)$$

and

$$\Delta X(n)+\frac{q_2(n)}{q_5^{\frac{\alpha_2}{\alpha_5}}(n-m+k+1)}(kc\theta_0)^{\frac{\alpha_2}{\alpha_5}}[X^{\frac{1}{\alpha_1}}(n-m+k+k_3-2) \qquad (27)$$
$$Q_1(n-m+k+k_3-1,n-m+k+k_3-1)]^{\frac{\alpha_2}{\alpha_5}}=0.$$

are oscillatory, then so is “Equation (1)”.

Proof:

Assume that y is a nonoscillatory solution of “Equation (1)”, say y(n) >0 , y(n-k) >0 ,
y(n-m+1) >0 , y(n+m*) >0 eventually.

It follows from “Equation (1)” that, eventually

$$\Delta(q_1(n)(\Delta^3 z(n))^{\alpha_1})=q_2(n)y^{\alpha_2}(n-m+1)+q_3(n)y^{\alpha_3}(n+m^*)>0 \qquad (28)$$

Therefore, “Equation (5)” is satisfied, and so, by Lemma 2.3 only four cases (I), (II), (III) and (IV) can be done.

Cases (I) and (II):

Using $\lambda=\alpha_5>1$, X = $q_5^{\frac{1}{\alpha_5}}(n)y(n-k)$, Y=$\left(\frac{1}{\alpha_5}q(n)q_5^{\frac{-1}{\alpha_5}}(n)\right)^{\frac{1}{\alpha_5-1}}$ in (3),

we get $q(n)y(n-k)-q_5(n)y^{\alpha_5}(n-k)\le g_2(n)$,

while using $\lambda=\alpha_4\in(0,1)$, X = $q_4^{\frac{1}{\alpha_4}}(n)y(n-k)$, Y=$\left(\frac{1}{\alpha_4}q(n)q_4^{\frac{-1}{\alpha_4}}(n)\right)^{\frac{1}{\alpha_4-1}}$ in (4),

we obtain $-(q(n)y(n-k)-q_4(n)y^{\alpha_4}(n-k))\le g_1(n)$.

Applying these two inequalities, we get

$$y(n)=z(n)-(q(n)\,y(n-k)-q_5(n)y^{\alpha_5}(n-k))+(q(n)\,y(n-k)-q_4(n)y^{\alpha_4}(n-k))$$
$$\ge z(n)-q_1(n)-q_2(n)$$
$$=\left(1-\frac{q_1(n)+q_2(n)}{z(n)}\right)z(n)$$

As z in both cases (I) and (II) is positive and non-decreasing, there exists L >0 fulfilling z(n) $>$ L, and thus, we obtain y(n) $\ge\left(1-\frac{q_1(n)+q_2(n)}{L}\right)z(n)$.

Then, due to “Equation (23)”, there exists k ∈ (0, 1) such that y(n) ≥ k z(n) eventually. (29)

So, we get

$$\Delta\left(q_1(n)(\Delta^3 z(n))^{\alpha_1}\right)\ge k^{\alpha_2}q_2(n)z^{\alpha_2}(n-m+1)+k^{\alpha_3}q_3(n)z^{\alpha_3}(n+m^*)\ge 0. \qquad (30)$$

Case (I):

Using “Equation (30)”, we get

$$\Delta\left(q_1(n)(\Delta^3 z(n))^{\alpha_1}\right)\ge k^{\alpha_3}q_3(n)z^{\alpha_3}(n+m^*). \qquad (31)$$

Summing "Equation (31)" from n-k_0 to n-1, we obtain

$$q_1(n)(\Delta^3 z(n))^{\alpha_1} \geq q_1(n-k_0)(\Delta^3 z(n-k_0))^{\alpha_1} + \sum_{r=n-k_0}^{n-1} \Delta(q_1(r)(\Delta^3 z(r))^{\alpha_1})$$

$$\geq k^{\alpha_3} \sum_{r=n-k_0}^{n-1} q_3(r) z^{\alpha_3}(r+m^*)$$

$$\geq k^{\alpha_3} z^{\alpha_3}(n+m^*-k_0) \sum_{r=n-k_0}^{n-1} q_3(r).$$

So, we get

$$\Delta^3 z(n) \geq k^{\frac{\alpha_3}{\alpha_1}} z^{\frac{\alpha_3}{\alpha_1}}(n+m^*-k_0)\left(\frac{1}{q_1(n)}\sum_{r=n-k_0}^{n-1} q_3(r)\right)^{\frac{1}{\alpha_1}}. \quad (32)$$

Summing "Equation (32)", again from n-k_0 to n-1, we get

$$\Delta^2 z(n) \geq \Delta^2 z(n-k_0) + \sum_{r=n-k_0}^{n-1} \Delta^3 z(r)$$

$$\geq \sum_{r=n-k_0}^{n-1} k^{\frac{\alpha_3}{\alpha_1}} z^{\frac{\alpha_3}{\alpha_1}}(r+m^*-k_0)\left(\frac{1}{q_1(r)}\sum_{t=r-k_0}^{r-1} q_3(t)\right)^{\frac{1}{\alpha_1}}$$

$$\geq k^{\frac{\alpha_3}{\alpha_1}} z^{\frac{\alpha_3}{\alpha_1}}(n+m^*-2k_0)\sum_{r=n-k_0}^{n-1}\left(\frac{1}{q_1(r)}\sum_{t=r-k_0}^{r-1} q_3(t)\right)^{\frac{1}{\alpha_1}}. \quad (33)$$

Summing "Equation (33)", again from n-k_0 to n-1, we obtain

$$\Delta z(n) \geq \Delta z(n-k_0) + \sum_{r=n-k_0}^{n-1} \Delta^2 z(r)$$

$$\geq k^{\frac{\alpha_3}{\alpha_1}} \sum_{r=n-k_0}^{n-1} z^{\frac{\alpha_3}{\alpha_1}}(r+m^*-2k_0)\sum_{t=r-k_0}^{r-1}\left(\frac{1}{q_1(t)}\sum_{\xi=t-k_0}^{t-1} q_3(\xi)\right)^{\frac{1}{\alpha_1}}$$

$$\geq k^{\frac{\alpha_3}{\alpha_1}} z^{\frac{\alpha_3}{\alpha_1}}(n+m^*-3k_0)\sum_{r=n-k_0}^{n-1}\left(\sum_{t=r-k_0}^{r-1}\left(\frac{1}{q_1(t)}\sum_{\xi=t-k_0}^{t-1} q_3(\xi)\right)^{\frac{1}{\alpha_1}}\right).$$

Hence we conclude that, z is a positive and increasing solution of

$$\Delta z(n) - k^{\frac{\alpha_3}{\alpha_1}} z^{\frac{\alpha_3}{\alpha_1}}(n+m^*-3k_0)\sum_{r=n-k_0}^{n-1}\left(\sum_{t=r-k_0}^{r-1}\left(\frac{1}{q_1(t)}\sum_{\xi=t-k_0}^{t-1} q_3(\xi)\right)^{\frac{1}{\alpha_1}}\right) \geq 0.$$

while applying Lemma (2.1) II, "Equation (24)" also has an eventually positive solution, which contradicts.

Case (II):

Let X = $-q_1(\Delta^3 z)^{\alpha_1} > 0$ eventually. (34)

By "Equation (32)", we get

$$-\Delta X(n) \geq k^{\alpha_2} q_2(n) z^{\alpha_2}(n-m+1). \quad (35)$$

We know that, eventually,

$$z(n) = z(n_1) + \sum_{r=n_1}^{n-1} \Delta z(r) \geq \sum_{r=n_1}^{n-1} \Delta z(n_1)$$

$$= (n-n_1)\Delta z(n-1) = n\,\Delta z(n-1)\left(1-\frac{n_1}{n}\right).$$

Since $\frac{n_1}{n} \to 0 \; as \; n \to \infty$, there exists $\theta_1 \in (0,1)$ such that z(n) $\geq n\theta_1 \Delta z(n-1)$ eventually. (36)

Next, we have

$$\Delta z(n) = \Delta z(n_2) + \sum_{r=n_2}^{n-1} \Delta^2 z(r) \geq \sum_{r=n_2}^{n-1} \Delta^2 z(n-1)$$

$$\geq (n-n_2)\Delta^2 z(n-1) = n\Delta^2 z(n-1)\left(1-\frac{n_2}{n}\right).$$

Since $\frac{n_2}{n} \to 0$ as $n \to \infty$, there exists $\theta_2 \in (0,1)$ such that $\Delta z(n) \geq n\theta_2 \Delta^2 z(n-1)$ eventually. (37)

Now, we set a $= n - m - 1$, b = a+k_1 > a.

Then, from "Equation (34)" and "Equation (28)", we get, eventually

$$0 \leq \Delta^2 z(b) = \Delta^2 z(a) + \sum_{r=a}^{b-1} \Delta^3 z(r) = \Delta^2 z(a) - \sum_{r=a}^{b-1} \frac{X^{\frac{1}{\alpha_1}}(r)}{q_1^{\frac{1}{\alpha_1}}(r)}$$

$$\leq \Delta^2 z(a) - \sum_{r=a}^{b-1} \frac{X^{\frac{1}{\alpha_1}}(b-1)}{q_1^{\frac{1}{\alpha_1}}(r)}$$

$$= \Delta^2 z(a) - X^{\frac{1}{\alpha_1}}(b-1) Q_1(b,a).$$

and therefore $\Delta^2 z(a) \geq X^{\frac{1}{\alpha_1}}(b-1) Q_1(b,a)$. (38)

From "Equations (35-38)", we obtain,

$$\begin{aligned} -\Delta X(n) &\geq k^{\alpha_2} q_2(n) z^{\alpha_2}(n-m+1) \\ &\geq (\theta_1 k)^{\alpha_2} q_2(n) (n-m+1)^{\alpha_2} (\Delta z(n-m))^{\alpha_2} \\ &\geq (\theta_1 \theta_2 k)^{\alpha_2} q_2(n)(n-m+1)^{\alpha_2} (n-m)^{\alpha_2} \left(\Delta^2 z(n-m-1)\right)^{\alpha_2} \\ &\geq (\theta_1 \theta_2 k)^{\alpha_2} q_2(n) (n-m+1)^{\alpha_2} (n-m)^{\alpha_2} \\ &\qquad X^{\frac{\alpha_2}{\alpha_1}}(n-m+k_1-2)[Q_1(n-m+k_1-1, n-m-1)]^{\alpha_2}. \end{aligned}$$

Therefore, X is a positive and decreasing solution of

$$\Delta X(n) + (\theta_1 \theta_2 k)^{\alpha_2} q_2(n)(n-m+1)^{\alpha_2} (n-m)^{\alpha_2}$$
$$X^{\frac{\alpha_2}{\alpha_1}}(n-m+k_1-2)[Q_1(n-m+k_1-1, n-m-1)]^{\alpha_2} \leq 0.$$

while applying Lemma (2.1) I, "Equation (25)" also has an eventually positive solution, which contradicts.

Case (III) and Case (IV):

In the rest of the proof, let X be as in "Equation (34)".

Now, $z(n) = y(n) + q_4(n) y^{\alpha_4}(n-k) - q_5(n) y^{\alpha_5}(n-k)$

$\geq -q_5(n) y^{\alpha_5}(n-k)$ eventually.

Thus, $y(n-k) \geq -\left(\frac{z(n)}{q_5(n)}\right)^{\frac{1}{\alpha_5}}$. (39)

Hence, from "Equation (1)", we get, eventually

$$\begin{aligned} -\Delta X(n) &\geq q_2(n) y^{\alpha_2}(n-m+1) + q_3(n) y^{\alpha_3}(n+m^*) \\ &\geq q_2(n) y^{\alpha_2}(n-m+1) \\ &\geq q_2(n) \left[\frac{z(n-m+k+1)}{q_5(n-m+k+1)}\right]^{\frac{\alpha_2}{\alpha_5}} \\ &= -\frac{q_2(n)}{q_5^{\frac{\alpha_2}{\alpha_5}}(n-m+k+1)} z^{\frac{\alpha_2}{\alpha_5}}(n-m+k+1). \end{aligned} \quad (40)$$

Case (III):

From "Equation (34)" and "Equation (28)", we obtain,

$$\Delta^2 z(n) = \Delta^2 z(n_1) + \sum_{r=n_1}^{n-1} \Delta^3 z(r) = \Delta^2 z(n_1) - \sum_{r=n_1}^{n-1} \frac{X^{\frac{1}{\alpha_1}}(r)}{q_1^{\frac{1}{\alpha_1}}(r)}$$

$$\leq -\sum_{r=n_1}^{n-1} \frac{X^{\frac{1}{\alpha_1}}(n-1)}{q_1^{\frac{1}{\alpha_1}}(r)}$$

$$=-X^{\frac{1}{\alpha_1}}(n-1)Q_1(n,n_1) \text{ eventually,}$$

which implies from "Equation (28)", we get

$$\Delta z(n) = \Delta z(n_1) + \sum_{r=n_1}^{n-1} \Delta^2 z(r)$$
$$\leq \sum_{r=n_1}^{n-1} \Delta^2 z(r) \leq -\sum_{r=n_1}^{n-1} X^{\frac{1}{\alpha_1}}(n-1)Q_1(r,n_1)$$
$$\leq -X^{\frac{1}{\alpha_1}}(n-2)\sum_{r=n_1}^{n-1} Q_1(r,n_1) \text{ eventually.}$$

Therefore, from "Equation (28)", we obtain

$$z(n) = z(n_1) + \sum_{r=n_1}^{n-1} \Delta z(r) \leq \sum_{r=n_1}^{n-1} \Delta z(r)$$
$$\leq -\sum_{r=n_1}^{n-1}[X^{\frac{1}{\alpha_1}}(n-2)(\sum_{t=r_1}^{r-1} Q_1(t,n_1))]$$
$$\leq -X^{\frac{1}{\alpha_1}}(n-3)\left[\sum_{r=n_1}^{n-1}\left(\sum_{t=r_1}^{r-1} Q_1(t,n_1)\right)\right] \text{ eventually.}$$

And thus, from "Equation (40)", we get, eventually

$$-\Delta X(n) \geq -\frac{q_2(n)}{q_5^{\frac{\alpha_2}{\alpha_5}}(n-m+k+1)} z^{\frac{\alpha_2}{\alpha_5}}(n-m+k+1)$$
$$\geq \frac{q_2(n)}{q_5^{\frac{\alpha_2}{\alpha_5}}(n-m+k+1)}\left[X^{\frac{1}{\alpha_1}}(n-m+k-2)\left(\sum_{r=n_1}^{n-m+k}\left(\sum_{t=r_1}^{r-m+k} Q_1(t,n_1)\right)\right)\right]^{\frac{\alpha_2}{\alpha_5}}$$
$$\geq \frac{q_2(n)}{q_5^{\frac{\alpha_2}{\alpha_5}}(n-m+k+1)} X^{\frac{\alpha_2}{\alpha_1\alpha_5}}(n-m+k-2)\left[\sum_{r=n_1}^{n-m+k}\left(\sum_{t=r_1}^{r-m+k} Q_1(t,n_1)\right)\right]^{\frac{\alpha_2}{\alpha_5}}.$$

Hence, X is a positive and decreasing solution of

$$\Delta X(n) + \frac{q_2(n)}{q_5^{\frac{\alpha_2}{\alpha_5}}(n-m+k+1)} X^{\frac{\alpha_2}{\alpha_1\alpha_5}}(n-m+k-2)\left[\sum_{r=n_1}^{n-m+k}\left(\sum_{t=r_1}^{r-m+k} Q_1(t,n_1)\right)\right]^{\frac{\alpha_2}{\alpha_5}} \leq 0.$$

While applying Lemma (2.1) I, we see that "Equation (26)", also has as an eventually positive solution, which contradicts.

Case (IV):

Let $a = n-m+k+1$, $b = a+k_2 > a$, $c = a+k_3-1 > b-1$, $d = a+k_4-1 > c-1$.

First, we get, eventually

$$0 \geq z(b) = z(a) + \sum_{r=a}^{b-1} \Delta z(r)$$
$$\geq z(a) + \sum_{r=a}^{b-1} \Delta z(b-1)$$
$$\geq z(a) + (b-a)\Delta z(b-1)$$
$$-z(a) \geq (b-a)\Delta z(b-1). \tag{41}$$

Next, we obtain, eventually

$$0 \leq \Delta z(c) = \Delta z(b-1) + \sum_{r=b-1}^{c-1} \Delta^2 z(r) \leq \Delta z(b-1) + \sum_{r=b-1}^{c-1} \Delta^2 z(c-1)$$
$$\leq \Delta z(b-1) + (c-b+1)\Delta^2 z(c-1)$$
$$\leq \Delta z(b-1) + c\left(1-\frac{b-1}{c}\right)\Delta^2 z(c-1).$$

Since $\frac{b-1}{c} \to 0$ as $n \to \infty$, there exists $\theta_0 \in (0,1)$ such that

$$\Delta z(b-1) \geq -c\theta_0 \Delta^2 z(c-1) \text{ eventually.} \tag{42}$$

Then, from "Equation (34)" and "Equation (28)", we have

$$\Delta^2 z(d) = \Delta^2 z(c-1) + \sum_{r=c-1}^{d-1} \Delta^3 z(r)$$

$$\leq -\sum_{r=c-1}^{d-1} \frac{X^{\frac{1}{\alpha_1}}(r)}{q_1^{\frac{1}{\alpha_1}}(r)} \leq -X^{\frac{1}{\alpha_1}}(d-1)Q_1(d,c-1).$$

$$-\Delta^2 z(d) \geq X^{\frac{1}{\alpha_1}}(d-1)Q_1(d,c-1). \tag{43}$$

Thus, from “Equations (40-43)”, we get

$$-\Delta X(n) \geq \frac{q_2(n)}{q_5^{\frac{\alpha_2}{\alpha_5}}(n-m+k+1)} [(b-a)\Delta z(b-1)]^{\frac{\alpha_2}{\alpha_5}}$$

$$\geq \frac{q_2(n)}{q_5^{\frac{\alpha_2}{\alpha_5}}(n-m+k+1)} k_2^{\frac{\alpha_2}{\alpha_5}}[-c\theta_0\Delta^2 z(c-1)]^{\frac{\alpha_2}{\alpha_5}}$$

$$\geq \frac{q_2(n)}{q_5^{\frac{\alpha_2}{\alpha_5}}(n-m+k+1)} (k_2 c\theta_0)^{\frac{\alpha_2}{\alpha_5}} \left[X^{\frac{1}{\alpha_1}}(c-2)Q_1(c-1,c-1)\right]^{\frac{\alpha_2}{\alpha_5}},$$

which implies, X is a positive and decreasing solution of

$$\Delta X(n) + \frac{q_2(n)}{q_5^{\frac{\alpha_2}{\alpha_5}}(n-m+k+1)} (k_2 c\theta_0)^{\frac{\alpha_2}{\alpha_5}} \left[X^{\frac{1}{\alpha_1}}(c-2)Q_1(c-1,c-1)\right]^{\frac{\alpha_2}{\alpha_5}} \leq 0$$

$$\Delta X(n) + \frac{q_2(n)}{q_5^{\frac{\alpha_2}{\alpha_5}}(n-m+k+1)} (kc\theta_0)^{\frac{\alpha_2}{\alpha_5}} [X^{\frac{1}{\alpha_1}}(n-m+k+k_3-2)$$

$$Q_1(n-m+k+k_3-1, n-m+k+k_3-1)]^{\frac{\alpha_2}{\alpha_5}} \leq 0.$$

Using Lemma (2.1) I, “Equation (27)” also has an eventually positive solution, which contradicts. ■

To illustrate Theorem 3.1, we have the following results.

Corollary 3.2

Consider that (i)-(iii), “Equation (2)”, “Equation (22)” and “Equation (23)” hold. If the first order advanced difference equation “Equation (24)” and the first order delay difference equations “Equation (25)” and

$$\Delta X(n) + \min\left(\begin{array}{l} \dfrac{q_2(n)}{q_5^{\frac{\alpha_2}{\alpha_5}}(n-m+k+1)} \displaystyle\sum_{r=n_1}^{n-m+k} \left(\sum_{t=r_1}^{r-m+k} Q_1(t,n_1)\right)^{\frac{\alpha_2}{\alpha_5}}, \\ \dfrac{q_2(n)}{q_5^{\frac{\alpha_2}{\alpha_5}}(n-m+k+1)} (kc\theta_0)^{\frac{\alpha_2}{\alpha_5}} [Q_1(n-m+k+k_3-1, n-m+k+k_3-1)] \end{array}\right) \tag{44}$$

$$X^{\frac{\alpha_2}{\alpha_1\alpha_5}}(n-m+k+k_3-2) = 0.$$

are oscillatory, for $\alpha_5 > 1 > \alpha_4$ and $\theta_0, \theta_1, \theta_2 \in (0,1)$, then “Equation (1)” is also oscillatory.

Corollary 3.3

Let $\alpha_5 > 1 > \alpha_4$ and $\alpha_3 \geq \alpha_1 \geq \alpha_2$. Suppose that (i)-(iii), “Equation (2)”, “Equation (22)” and “Equation (23)” hold. If

$$\lim_{n\to\infty} sup \sum_{r=n-k_0}^{n-1} \left(\sum_{t=r-k_0}^{r-1} \left(\frac{1}{q_1(t)} \sum_{\xi=t-k_0}^{t-1} q_3(\xi)\right)^{\frac{1}{\alpha_1}}\right) = \infty, \tag{45}$$

$$\lim_{n\to\infty} \sup q_2(n)(n-m+1)^{\alpha_2}(n-m)^{\alpha_2} Q_1[(n-m+k_1-1, n-m-1)]^{\alpha_2} = \infty, \tag{46}$$

$$\lim_{n\to\infty} sup \frac{q_2(n)}{q_5^{\frac{\alpha_2}{\alpha_5}}(n-m+k+1)}\left[\sum_{r=n_1}^{n-m+k}\left(\sum_{t=r_1}^{r-m+k} Q_1(t,n_1)\right)\right]^{\frac{\alpha_2}{\alpha_5}} = \infty, \tag{47}$$

and

$$\lim_{n\to\infty} sup \frac{q_2(n)}{q_5^{\frac{\alpha_2}{\alpha_5}}(n-m+k+1)}(k_2 c\theta_0)^{\frac{\alpha_2}{\alpha_5}}\left[\begin{matrix} Q_1(n-m+k+k_3-1, \\ n-m+k+k_3-1)\end{matrix}\right]^{\frac{\alpha_2}{\alpha_5}} = \infty, \tag{48}$$

then "Equation (1)" is oscillatory.

Corollary 3.4

Let $1 \geq \alpha_5 > \alpha_4$. Suppose that (i)-(iii), "Equation (2)" and "Equation (22)" hold.

Assume $\lim_{n\to\infty} Q(n) = 0$, where $Q(n) = \frac{\alpha_5-\alpha_4}{\alpha_4}\left(\frac{\alpha_4}{\alpha_5} q_4(n)\right)^{\frac{\alpha_5}{\alpha_5-\alpha_4}}(q_5(n))^{\frac{\alpha_4}{\alpha_4-\alpha_5}}$. (49)

Let $\theta_0, \theta_1, \theta_2 \in (0,1)$. If "Equations (24-27)" are oscillatory, then so is "Equation (1)".

Corollary 3.5 Let $1 \geq \alpha_5 > \alpha_4$. Assume that (i)-(iii), "Equation (2)", "Equation (22)" and "Equation (49)" hold. Let $\theta_0, \theta_1, \theta_2 \in (0,1)$. If "Equation (24)", "Equation (25)" and "Equation (44)" are oscillatory, then "Equations (1)" is oscillatory.

Corollary 3.6 Let $1 \geq \alpha_5 > \alpha_4 \; and \; \alpha_3 \geq \alpha_1 \geq \alpha_2/\alpha_5$. Assume that (i)-(iii), "Equation (2)", "Equation (22)" and "Equation (49)" hold. Let $\theta_0, \theta_1, \theta_2 \in (0,1)$. If "Equations (45-48)" hold, then "Equation (1)" is oscillatory.

4 EXAMPLES

The following examples are illustrative:

Example 4.1 We consider the equation

$$\Delta\left((n+2)^3\left(\Delta^3\left(y(n)+\frac{1}{n}y^{\frac{1}{3}}(n-1)-y^3(n-1)\right)\right)^3\right)$$
$$= n^2 y(n-3) + (n+3)^4 y(n+6). \tag{50}$$

Now "Equation (50)" is in the form "Equation (1)", where

$\alpha_1 = \alpha_3 = \alpha_5 = 3,\; \alpha_2 = 1, \alpha_4 = \frac{1}{3}, k = 1, m = 4, m^* = 6.$

$q_1(n) = (n+2)^3, q_2(n) = n^2, q_3(n) = (n+3)^4, q_4(n) = \frac{1}{n}, q_5(n) = 1.$

Then, (i)-(iii) are fulfilled, and so is (2), because

$$Q_1(b,a) = \sum_{r=a}^{b-1}\left(\frac{1}{q_1(r)}\right)^{\frac{1}{3}} = \sum_{r=a}^{b-1}\frac{1}{r+1} = \sum_{r=a+1}^{b}\frac{1}{r} \to \infty.$$

Now we may pick (see Note 2.4)

$k_0 = k_1 = k_2 = 1$ and $k_3 = 2$.

and thus "Equation (22)" is fulfilled, and we get

$n + m^* - 3k_0 = n + 3.$

$n - m + k_1 - 2 = n - m + k - 2 = n - 5.$

$n - m + k - 2 + k_3 = n - 3.$

Furthermore, we have $\alpha_5 > 1 > \alpha_4$ and $\alpha_2 < \alpha_1 = \alpha_3$, so by Corollary 3.3. We choose $q = q_4$, and then

$$g_1(n) + g_2(n) = \frac{2}{3}\left(\frac{1}{3}\right)^{\frac{1}{2}}(q(n))^{\frac{-1}{2}}(q_4(n))^{\frac{3}{2}} + 2.3^{\frac{-3}{2}}(q(n))^{\frac{3}{2}}((q_5(n))^{\frac{-1}{2}}$$
$$= \frac{2}{3\sqrt{3}n}\left(1+\frac{1}{\sqrt{n}}\right) \to 0 \text{ as } n \to \infty.$$

And so, "Equation (23)" is fulfilled. We also compute

$$\sum_{r=n-k_0}^{n-1}\left(\sum_{t=r-k_0}^{r-1}\left(\frac{1}{q_1(t)}\sum_{\xi=t-k_0}^{t-1}q_3(\xi)\right)^{\frac{1}{\alpha_1}}\right)=\left(\frac{q_3(n-3)}{q_1(n-1)}\right)^{\frac{1}{3}}=n^{\frac{1}{3}},$$

$$q_2(n)(n-m+1)^{\alpha_2}(n-m)^{\alpha_2}[Q_1(n-m+k_1-1,n-m-1)]^{\alpha_2}$$

$$=q_2(n)(n-3)(n-4)[Q_1(n-4,n-5)]=n^2(n-4),$$

$$\frac{q_2(n)}{q_5^{\frac{\alpha_2}{\alpha_5}}(n-m+k+1)}\left[\sum_{r=n_1}^{n-m+k}\left(\sum_{t=r_1}^{r-m+k}Q_1(t,n_1)\right)\right]^{\frac{\alpha_2}{\alpha_5}}\geq\frac{n^2}{(n-5)^{\frac{1}{3}}}\geq n^{\frac{5}{3}},$$

and

$$\frac{q_2(n)}{q_5^{\frac{\alpha_2}{\alpha_5}}(n-m+k+1)}[Q_1(n-m+k+k_3-1,n-m+k+k_3-1)]^{\frac{\alpha_2}{\alpha_5}}$$

$$=q_2(n)[Q_1(n-2,n-2)]^{\frac{1}{3}}=q_2(n)\left(\frac{2n-1}{n(n-1)}\right)^{\frac{1}{3}}=(2)^{\frac{1}{3}}n^2.$$

Hence, "Equations (45-48)" hold. Now all the conditions of Corollary 3.3 are satisfied, and so "Equation (50)" is oscillatory.

Example 4.2 We consider the equation

$$\Delta\left((n+2)^3\left(\Delta^3\left(y(n)+\frac{1}{n}y^{\frac{1}{3}}(n-1)-y^{\frac{2}{3}}(n-1)\right)\right)^3\right)$$

$$=n^2y(n-3)+(n+3)^4y(n+6). \tag{51}$$

In the above equation all the statistics are the similar as in "Equation (50)", excluding $\alpha_5=\frac{2}{3}$.

Furthermore, we have $1>\alpha_5>\alpha_4$ and $\alpha_3=\alpha_1>\alpha_2/\alpha_5$, from Corollary 3.6.

We compute $Q(n)=\left(\frac{1}{2}q_4(n)\right)^2(q_5(n))^{-1}=\frac{1}{4n^2}\to 0$ as $n\to\infty$.

and so "Equation (49)" is fulfilled. All other conditions of Corollary 3.6 are satisfied in the similar way as in Example 4.1, and therefore "Equation (51)" is oscillatory.

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BIOGRAPHY

Dr. S. Kaleeswari was born in Pollachi, Tamil Nadu, India. She received her B.Sc., degree in Mathematics from NGM College, Bharathiar University, Coimbatore in 1999, her M.Sc., degree in Mathematics from NGM College, Bharathiar University, Coimbatore in 2001, her M.Phil., degree in Mathematics from Bharathiar University, Coimbatore in 2003 and her Ph.D degree in Mathematics from Anna University, Chennai in 2017. She is doing her research in the field of difference and differential equations.

She is an Assistant Professor in the Department of Mathematics, Nallamuthu Gounder Mahalingam College, Pollachi. She has 16 years of experience in teaching and She has published 15 papers in national and international journals and one book chapter. She published her results on oscillation theory of ordinary and delay difference equations. She has life membership in Indian mathematical society, Ramanujan mathematical society, Indian science congress association. She acts as reviewer for some refereed journals.

Ms. M. Buvanasankari was born in Sattur, Tamil Nadu, India. She received her B.Sc., degree in Mathematics from SRNM College, MK University, Madurai in 2006, her M.Sc., degree in Mathematics from SRNM College, MK University, Madurai in 2008, her M.Phil., degree in Mathematics from MK University, Madurai in 2009, during her Ph.D., degree in Mathematics from NGM College, Bharathiar University, Coimbatore in 2021. She is doing her research in the field of difference and differential equations.

She is an Assistant Professor(SG) in the Department of Mathematics, Nehru Institute of Engineering and Technology, Coimbatore. She has 12 years of experience in teaching and She has life membership in IAENG.