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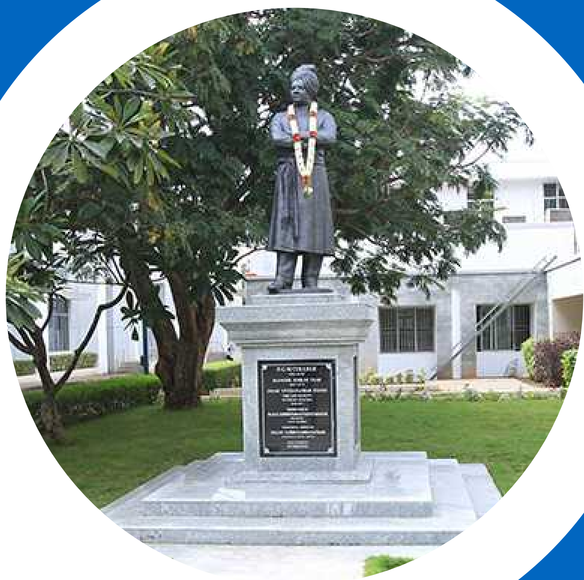

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27th October 2021

Jointly Organized by

Department of Biological Science, Physical Science and Computational Science

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ABOUT THE INSTITUTION

A nations's growth is in proportion to education and intelligence spread among the masses. Having this idealistic vision, two great philanthropists late. S.P. Nallamuthu Gounder and Late. Arutchelver Padmabhushan Dr.N.Mahalingam formed an organization called Pollachi Kalvi Kazhagam, which started NGM College in 1957, to impart holistic education with an objective to cater to the higher educational needs of those who wish to aspire for excellence in knowledge and values. The College has achieved greater academic distinctions with the introduction of autonomous system from the academic year 1987-88. The college has been Re-Accredited by NAAC and it is ISO 9001 : 2015 Certified Institution. The total student strength is around 6000. Having celebrated its Diamond Jubilee in 2017, the college has blossomed into a premier Post-Graduate and Research Institution, offering 26 UG, 12 PG, 13 M.Phil and 10 Ph.D Programmes, apart from Diploma and Certificate Courses. The college has been ranked within Top 100 (72nd Rank) in India by NIRF 2021.

ABOUT CONFERENCE

The International conference on “Emerging Trends in Science and Technology (ETIST-2021)” is being jointly organized by Departments of Biological Science, Physical Science and Computational Science - Nallamuthu Gounder Mahalingam College, Pollachi along with ISTE, CSI, IETE, IEE & RIYASA LABS on 27th OCT 2021. The Conference will provide common platform for faculties, research scholars, industrialists to exchange and discus the innovative ideas and will promote to work in interdisciplinary mode.

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Third order Nonlinear Difference Equations with a Superlinear Neutral term

S. Kaleeswari[1] and **Ercan Tunc**[2]

Abstract - This paper deals with the oscillatroy and asymptotic behavior of third order nonlinear difference equations with a superlinear neutral term. Sufficient conditions which improve, extend and simplify existing once in the literature are presented. Two examples are provided in order to illustrate the significance of the main results.

Keywords *Convergence, Nonlinear, Neutral term, Oscillations, Third order difference equations.*
2010 Subject classification: *39A10.*

1 Introduction

This paper is concerned with oscillatory and asymptotic results for all solutions of the third order nonlinear difference equations with a superlinear neutral term of the form

$$\Delta\left[a(t)\left(\Delta^2 y(t)\right)^{\alpha}\right]+c(t)x^{\gamma}\left(\sigma(t)\right)=0, \qquad t\geq t_0>0, \tag{1}$$

where $y(t)=x(t)+b(t)x^{\beta}\left(\zeta(t)\right)$ and $t\in N=\{t_0,t_0+1,\ldots\}$, t_0 is a positive integer.

We assume that

(H1) $\{a(t)\}$, $\{b(t)\}$ and $\{c(t)\}$ are real sequences with $a(t)>0$, $b(t)\geq 1$, $b(t)\neq 1$ for large t, $c(t)\geq 0$ and $q(t)$ is not identically zero for large t;

(H2) $\{\zeta(t)\}$, $\{\sigma(t)\}$ are real sequences such that $\zeta(t)<t$, $\sigma(t)<t$, ζ is strictly increasing, $\sigma(t)$ is nondecreasing and $\lim_{t\to\infty}\zeta(t)=\lim_{t\to\infty}\sigma(t)=\infty$;

(H3) α,β and γ are the ratios of odd positive integers with $\beta\geq 1$;

(H4) $h(t)=\zeta^{-1}\left(\sigma(t)\right)\leq t-1$ and $\lim_{t\to\infty}h(t)=\infty$.

We let

$$S_1(v,u)=\sum_{s=u}^{v-1}a^{-\frac{1}{\alpha}}(s), \qquad v\geq u\geq t_0$$

and assume that

$$S_1(t,t_0)\to\infty \,\text{as}\, t\to\infty. \tag{2}$$

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The real sequence $\{x(t)\}$ is said to be solution of (1.1) if it is defined and satisfies (1.1) for all $t \in N(t_0)$. We only consider those solutions of (1.1) that satisfy $sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq t_0$ and we tacitly assume that (1) possesses such solutions. A solution of (1) is called oscillatory if it is neither eventually positive nor eventually negative and it is called nonoscillatory otherwise. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

For the basic theory of difference equations and its applications, one can refer the monographs by Agarwal [1, 2], the papers [19-24] and reference cited therein. In recent years, numerous researchers have analyzed the asymptotic and oscillatory behavior of solutions to various classes of neutral difference equations; see the papers [6-8, 10-14]. This is due to the fact that such equations find numerous applications in natural sciences and technology. For instance, the equations of this type appear in the study of electric networks, vibrating masses attached to an elastic bar and in the solution of variational problems with time delays (see [9, 15-18]).

To the best of our knowledge, there are no papers at the present time dealing with third order difference equations with superlinear neutral term. Motivated by these observations, the aim of this paper is to obtain sufficient conditions under which every solution of equation (1) is either oscillates or converges to zero as $t \to \infty$.

2 Main Results

For simplicity, we set

$$S_2(t, t_2) = \sum_{s=t_2}^{t-1} S_1(s, t_1), \qquad t \geq t_2 \geq t_1,$$

where $t_1 \geq t_0$.
Throughout this paper, we assume that,

$$P_1(t) = \frac{1}{b\left(\zeta^{-1}(t)\right)} \left[1 - \left(\frac{S_2\left(\zeta^{-1}\left(\zeta^{-1}(t)\right), t_2\right)}{S_2\left(\zeta^{-1}(t), t_2\right)}\right)^{\frac{1}{\beta}} \frac{k^{\frac{1}{\beta}-1}}{b^{\frac{1}{\beta}}\left(\zeta^{-1}\left(\zeta^{-1}(t)\right)\right)}\right] \geq 0 \tag{3}$$

and

$$P_2(t) = \frac{1}{b\left(\zeta^{-1}(t)\right)} \left(1 - \frac{l^{\frac{1}{\beta}-1}}{b^{\frac{1}{\beta}}\left(\zeta^{-1}\left(\zeta^{-1}(t)\right)\right)}\right) \geq 0 \tag{4}$$

for all sufficiently large t and for every positive constants k and l.

Remark 2.1. *Since*

$$\begin{aligned} P(t, t_2) &= \frac{S_2\left(\zeta^{-1}(t), t_2\right)}{S_2(t, t_2)} \frac{1}{b(\zeta^{-1}(t))} \\ &\geq \frac{1}{b(\zeta^{-1}(t))}, \end{aligned}$$

then the condition

$$\lim_{t\to\infty} P(t, t_2) = 0; \qquad \beta > 1$$

$$\lim_{t\to\infty} P(t,t_2) < 1; \qquad \beta = 1$$

ensures the positivity of the sequences P_1 *and* P_2.

Lemma 2.2. *Assume that the conditions* $[H_1]$-$[H_3]$ *and* (2) *hold and let* $x(t)$ *is an eventually positive solution of equation* (1). *Then there exists* $t_1 \geq t_0$ *such that the sequence* $\{y(t)\}$ *satisfies one of the following two cases:*

(I) $y(t) > 0, \Delta y(t) > 0, \Delta^2 y(t) > 0$ *and* $\Delta\left(a(t)\left(\Delta^2 y(t)\right)^{\alpha}\right) \leq 0$

(I) $y(t) > 0, \Delta y(t) < 0, \Delta^2 y(t) > 0$ *and* $\Delta\left(a(t)\left(\Delta^2 y(t)\right)^{\alpha}\right) \leq 0$

for $t \geq t_1$.

Proof. The proof is immediate. Hence we omit the details. □

Lemma 2.3. *Suppose that the conditions* $[H_1]$-$[H_4]$ *and* (2) *hold. Let* $x(t)$ *be an eventually positive solution of equation* (1) *with* $y(t)$ *satisfying case (I) of Lemma 2.2. Then* $y(t)$ *satisfies*

$$\Delta\left(a(t)\left(\Delta^2 y(t)\right)^{\alpha}\right) + c(t) P_1^{\frac{\gamma}{\beta}}\left(\sigma(t)\right) y^{\frac{\gamma}{\beta}}\left(h(t)\right) \leq 0 \tag{5}$$

for large t.

Proof. Assume that $x(t)$ is an eventually positive solution of (1), say $x(t) > 0$, $x\left(\zeta(t)\right) > 0$ and $x\left(\sigma(t)\right) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. From the definition of $y(t)$, we have

$$x^{\beta}\left(\zeta(t)\right) = \frac{1}{b(t)}\left(y(t) - x(t)\right) \leq \frac{y(t)}{b(t)}.$$

Since $\zeta(t) < t$ is strictly increasing, we can see that

$$x\left(\zeta^{-1}(t)\right) \leq \frac{y^{\frac{1}{\beta}}\left(\zeta^{-1}\left(\zeta^{-1}(t)\right)\right)}{b^{\frac{1}{\beta}}\left(\zeta^{-1}\left(\zeta^{-1}(t)\right)\right)}.$$

Using the above inequality in the definition of $y(t)$ gives

$$\begin{aligned} x^{\beta}(t) &= \frac{1}{b\left(\zeta^{-1}(t)\right)}\left[y\left(\zeta^{-1}(t)\right) - x\left(\zeta^{-1}(t)\right)\right] \\ &\geq \frac{1}{b\left(\zeta^{-1}(t)\right)}\left[y\left(\zeta^{-1}(t)\right) - \frac{y^{\frac{1}{\beta}}\left(\zeta^{-1}\left(\zeta^{-1}(t)\right)\right)}{b^{\frac{1}{\beta}}\left(\zeta^{-1}\left(\zeta^{-1}(t)\right)\right)}\right]. \end{aligned} \tag{6}$$

Since $a(t)\left(\Delta^2 y(t)\right)^{\alpha}$ is nonincreasing for $t \geq t_1$, we obtain

$$\begin{aligned} \Delta y(t) &= \Delta y(t_1) + \sum_{s=t_1}^{t-1} \frac{\left(a(s)\left(\Delta^2 y(s)\right)^{\alpha}\right)^{\frac{1}{\alpha}}}{a^{\frac{1}{\alpha}}(s)} \\ &\geq \left(a(t)\left(\Delta^2 y(t)\right)^{\alpha}\right)^{\frac{1}{\alpha}} S_1(t,t_1). \end{aligned} \tag{7}$$

From (7), for all $t \geq t_2 = t_1 + 1$, we have

$$\Delta\left(\frac{\Delta y(t)}{S_1(t,t_1)}\right) = \frac{a^{\frac{-1}{\alpha}}(t)\left[a^{\frac{1}{\alpha}}(t)\Delta^2 y(t) S_1(t,t_1) - \Delta y(t)\right]}{S_1(t,t_1)S_1(t+1,t_1)} \leq 0.$$

i.e. $\frac{\Delta y(t)}{S_1(t,t_1)}$ is nonincreasing for $t \geq t_2$. Hence we obtain

$$\begin{aligned} y(t) &= y(t_2) + \sum_{s=t_2}^{t-1} \frac{\Delta y(t)}{S_1(s,t_1)} S_1(s,t_1) \\ &\geq \frac{\Delta y(t)}{S_1(t,t_1)} \sum_{s=t_2}^{t-1} S_1(s,t_1) \\ &= \frac{S_2(t,t_2)}{S_1(t,t_1)} \Delta y(t), \qquad t \geq t_2. \end{aligned}$$

Thus for all $t \geq t_3 = t_2 + 1$, we have

$$\Delta\left[\frac{y(t)}{S_2(t,t_2)}\right] = \frac{\Delta y(t) S_2(t,t_2) - y(t) S_1(t,t_1)}{S_2(t,t_2)S_2(t+1,t_2)} \leq 0.$$

i.e. $\frac{y(t)}{S_2(t,t_2)}$ is nonincreasing for $t \geq t_3$. Since $\zeta(t) < t$ and ζ is strictly increasing, we can see that ζ^{-1} is increasing and $t < \zeta^{-1}(t)$. Thus we obtain

$$\zeta^{-1}(t) \leq \zeta^{-1}\left(\zeta^{-1}(t)\right). \tag{8}$$

Since $\frac{y(t)}{S_2(t,t_2)}$ is nonincreasing, from (8), we get

$$\frac{S_2\left(\zeta^{-1}\left(\zeta^{-1}(t)\right), t_2\right) y\left(\zeta^{-1}(t)\right)}{S_2\left(\zeta^{-1}(t), t_2\right)} \geq y\left(\zeta^{-1}\left(\zeta^{-1}(t)\right)\right).$$

Using this in (2.4) gives

$$x^\beta(t) \geq \frac{y\left(\zeta^{-1}(t)\right)}{b\left(\zeta^{-1}(t)\right)}\left[1 - \left(\frac{S_2\left(\zeta^{-1}\left(\zeta^{-1}(t)\right), t_2\right)}{S_2\left(\zeta^{-1}(t), t_2\right)}\right)^{\frac{1}{\beta}} \frac{y^{\frac{1}{\beta}-1}\left(\zeta^{-1}(t)\right)}{b^{\frac{1}{\beta}}\left(\zeta^{-1}\left(\zeta^{-1}(t)\right)\right)}\right], \tag{9}$$

for $t \geq t_3$. Since $y(t)$ is positive and increasing for $t \geq t_3$, we can find $t_4 \geq t_3$ and a constant $k > 0$ such that

$$y(t) \geq k \text{ for } t \geq t_4. \tag{10}$$

From (9) and (10), we have

$$\begin{aligned} x^\beta(t) &\geq \frac{y\left(\zeta^{-1}(t)\right)}{b\left(\zeta^{-1}(t)\right)}\left[1 - \left(\frac{S_2\left(\zeta^{-1}\left(\zeta^{-1}(t)\right), t_2\right)}{S_2\left(\zeta^{-1}(t), t_2\right)}\right)^{\frac{1}{\beta}} \frac{k^{\frac{1}{\beta}-1}}{b^{\frac{1}{\beta}}\left(\zeta^{-1}\left(\zeta^{-1}(t)\right)\right)}\right] \\ &= P_1(t) y(\zeta^{-1}(t)), \qquad t \geq t_4. \end{aligned}$$

So,

$$x^{\beta}(\sigma(t)) \geq P_1(\sigma(t))\, y\left(\zeta^{-1}(\sigma(t))\right) \text{ for } t \geq t_5;$$

where $\sigma(t) \geq t_4$ for $t \geq t_5$ for some $t_5 \geq t_4$. Substituting this in (1) gives

$$\Delta\left(a(t)\left(\Delta^2 y(t)\right)^{\alpha}\right) \leq -c(t) P_1^{\frac{\gamma}{\beta}}(\sigma(t))\, y^{\frac{\gamma}{\beta}}(h(t)) \text{ for } t \geq t_5. \tag{11}$$

i.e., (5) holds and hence the proof. □

Lemma 2.4. *Let conditions* $[H_1]$-$[H_4]$ *and* (2) *hold and let* $x(t)$ *be an eventually positive solution of equation* (1) *with* $y(t)$ *satisfying case (II) of Lemma 2.2. Then* $y(t)$ *either satisfies*

$$\Delta\left(a(t)\left(\Delta^2 y(t)\right)^{\alpha}\right) + c(t) P_2^{\frac{\gamma}{\beta}}(\sigma(t))\, y^{\frac{\gamma}{\beta}}(h(t)) \leq 0 \tag{12}$$

for large t *or* $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = 0$.

Proof. Suppose that $x(t)$ is an eventually positive solution of (1) such that $x(t) > 0$, $x(\zeta(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Proceeding as in the proof of Lemma 2.3, we can see that (6) and (8) hold. Since $\Delta y(t) < 0$, from (8), we obtain

$$y\left(\zeta^{-1}(t)\right) \geq y\left(\zeta^{-1}(\zeta^{-1}(t))\right).$$

Using the last inequality in (6) gives

$$x^{\beta}(t) \geq \frac{y(\zeta^{-1}(t))}{b(\zeta^{-1}(t))}\left[1 - \frac{y^{\frac{1}{\beta}-1}(\zeta^{-1}(t))}{b^{\frac{1}{\beta}}(\zeta^{-1}(\zeta^{-1}(t)))}\right]. \tag{13}$$

Since $y(t)$ satisfies case (II) of Lemma 2.2, there exists a constant l such that $\lim_{t\to\infty} y(t) = l < \infty$.
case(a): If $l > 0$, then there exists $t_2 \geq t_1$ such that

$$y(t) \geq l; \text{ for } t \geq t_2. \tag{14}$$

From (14), we obtain

$$y^{\frac{1}{\beta}-1}(t) \leq l^{\frac{1}{\beta}-1}.$$

Using this in (13) yields

$$x^{\beta}(t) \geq \frac{y(\zeta^{-1}(t))}{b(\zeta^{-1}(t))}\left[1 - \frac{l^{\frac{1}{\beta}-1}}{b^{\frac{1}{\beta}}(\zeta^{-1}(\zeta^{-1}(t)))}\right] = P_2(t) y(\zeta^{-1}(t)).$$

Therefore (1) becomes

$$\Delta\left(a(t)\left(\Delta^2 y(t)\right)^{\alpha}\right) \leq -c(t) P_2^{\frac{\gamma}{\beta}}(\sigma(t)) y^{\frac{\gamma}{\beta}}(h(t)), \tag{15}$$

for $t \geq t_3$ for some $t_3 \geq t_2$. i.e., (12) holds.
case(b): If $l = 0$, then $\lim_{t\to\infty} y(t) = 0$.
Since $0 < x(t) \leq y(t)$ for $t \geq t_1$, we have $\lim_{t\to\infty} x(t) = 0$. This completes the proof. □

Theorem 2.5. *Assume that conditions* $[H_1]$-$[H_4]$ *and* (2) *hold. If for all sufficiently large* $t_1 \geq t_0$ *and for some* $t_2 \geq t_1$,

$$\sum_{s=t_2}^{\infty} c(s) P_1^{\frac{\gamma}{\beta}}(\sigma(t)) = \infty \tag{16}$$

and

$$\sum_{s=t_0}^{\infty} c(s) P_2^{\frac{\gamma}{\beta}}(\sigma(t)) = \infty \tag{17}$$

then every solution $x(t)$ *of equation* (1) *is either oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. Suppose that $x(t)$ is a nonoscillatory solution of equation (1), say $x(t) > 0$, $x(\zeta(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Assume (3) and (4) hold for $t \geq t_1$. Then from Lemma 2.2, $y(t)$ satisfies either case (I) or case (II) for $t \geq t_1$.

First we consider case (I). From Lemma 2.3, we can see that inequalities (10) and (11) hold for $t \geq t_5$. Using (10) in (11) yields

$$\Delta\left(a(t)\left(\Delta^2 y(t)\right)^{\alpha}\right) \leq -k^{\frac{\gamma}{\beta}} c(t) P_1^{\frac{\gamma}{\beta}}(\sigma(t)); \text{ for } t \geq t_5. \tag{18}$$

Summing (18) from t_5 to $t-1$ gives

$$a(t)\left(\Delta^2 y(t)\right)^{\alpha} \leq a(t_5)\left(\Delta^2 y(t)\right)^{\alpha} - k^{\frac{\gamma}{\beta}} \sum_{s=t_5}^{t-1} c(s) P_1^{\frac{\gamma}{\beta}}(\sigma(t)) \to -\infty,$$

as $t \to \infty$. Which is a contradiction to the fact that $a(t)\left(\Delta^2 y(t)\right)^{\alpha}$ is positive.

Next we consider case (II). Then from Lemma 2.4, we again arive at case (a) or case (b). In case (a), (14) and (15) hold for $t \geq t_3$. Using (14) in (15) gives

$$\Delta\left(a(t)\left(\Delta^2 y(t)\right)^{\alpha}\right) \leq -l^{\frac{\gamma}{\beta}} c(t) P_2^{\frac{\gamma}{\beta}}(\sigma(t)); \qquad t \geq t_3. \tag{19}$$

Summing (19) from t_3 to $t-1$ gives

$$a(t)\left(\Delta^2 y(t)\right)^{\alpha} \leq a(t_3)\left(\Delta^2 y(t_3)\right)^{\alpha} - l^{\frac{\gamma}{\beta}} \sum_{s=t_3}^{t-1} c(s) P_2^{\frac{\gamma}{\beta}}(\sigma(s)) \to -\infty,$$

as $t \to \infty$. Which again contradicts the fact that $a(t)$ $\left(\Delta^2 y(t)\right)^{\alpha}$ is positive. In case (b), as in Lemma 2.4, $y(t) \to 0$ as $t \to \infty$. This completes the proof. □

Theorem 2.6. *Suppose that conditions* $[H_1]$-$[H_4]$ *and* (2) *hold. Assume that there exist real sequences* $\{\eta(t)\}$ *and* $\{\varsigma(t)\}$ *such that* $h(t) \leq \eta(t) \leq \varsigma(t) \leq t-1$ *for* $t \geq t_0$. *If the first order delay difference equations*

$$\Delta z(t) + c(t) P_1^{\frac{\gamma}{\beta}}(\sigma(t)) S_2^{\frac{\gamma}{\beta}}\left(h(t), t_0\right) z^{\frac{\gamma}{\alpha\beta}}(h(t)) = 0 \tag{20}$$

and

$$\Delta r(t) + c(t) P_2^{\frac{\gamma}{\beta}}(\sigma(t)) \left[\left(\eta(t) - h(t)\right) S_1\left(\varsigma(t), \eta(t)\right)\right]^{\frac{\gamma}{\beta}} r^{\frac{\gamma}{\alpha\beta}}\left(\varsigma(t)\right) = 0 \tag{21}$$

are oscillatory, then every solution $x(t)$ *of equation* (1) *is either oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1) say $x(t) > 0$, $x(\zeta(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$ and assume (3) and (4) hold for $t \geq t_1$. Then from Lemma 2.2, $x(t)$ satisfies either case (I) or case (II) for $t \geq t_1$.

First we consider case (I), proceeding as in the proof of Lemma 2.3, we again arive at (7) for $t \geq t_1$ and (11) for $t \geq t_5$. Summing (7) from t_1 to $t-1$ gives

$$\begin{aligned} y(t) &\geq \sum_{s=t_1}^{t-1} S_1(s,t_1)\left(a(t)\left(\Delta^2 y(t)\right)^{\alpha}\right)^{\frac{1}{\alpha}} \\ &= S_2(t,t_1)\left(a(t)\left(\Delta^2 y(t)\right)^{\alpha}\right)^{\frac{1}{\alpha}} \end{aligned}$$

and so

$$y(h(t)) \geq S_2\left(h(t),t_1\right)\left(a(h(t))\left(\Delta^2 y(h(t))\right)^{\alpha}\right)^{\frac{1}{\alpha}}, \qquad t \geq t_2,$$

where $h(t) \geq t_1$ for $t \geq t_2$ for some $t_2 \geq t_1$. Using this in (11), we can see that

$$\Delta\left(a(t)\left(\Delta^2 y(t)\right)^{\alpha}\right) + c(t)P_1^{\frac{\gamma}{\beta}}(\sigma(t))S_2^{\frac{\gamma}{\beta}}(h(t),t_1)\left(a(h(t))\left(\Delta^2 y(h(t))\right)^{\alpha}\right)^{\frac{\gamma}{\beta}} \leq 0 \tag{22}$$

for $t \geq t_5$.

Letting $z(t) = a(t)\left(\Delta^2 y(t)\right)^{\alpha}$, we see that $z(t)$ is a positive solution of the first order delay difference inequality

$$\Delta z(t) + c(t)P_1^{\frac{\gamma}{\beta}}(\sigma(t))S_2^{\frac{\gamma}{\beta}}\left(h(t),t_1\right) z^{\frac{\gamma}{\alpha\beta}}(h(t)) \leq 0. \tag{23}$$

The function $z(t)$ is decreasing for $t \geq t_5$ and so by a well-known result [19, Theorem 1], there exists a positive solution of equation (20) which is a contradiction to the fact that equation (20) is oscillatory.

Next we consider case (II). By Lemma 2.4, we again have case (a) or case (b). In case (a), we can see that (15) holds for $t \geq t_3$. Since case (II) holds, for $v \geq u \geq t_3$, we have

$$\begin{aligned} y(u) &= y(v) + \sum_{s=u}^{v-1}\left(-\Delta y(s)\right) \\ &\geq (v-u)\left(-\Delta y(v)\right). \end{aligned} \tag{24}$$

Setting $u = h(t)$ and $v = \eta(t)$ in (24), we get

$$y(h(t)) \geq (\eta(t) - h(t))\left(-\Delta y(\eta(t))\right). \tag{25}$$

Since $\Delta y(t) < 0$ and $a(t)\left(\Delta^2 y(t)\right)^{\alpha}$ is decreasing, we have

$$\begin{aligned} -\Delta y(u) &\geq \Delta y(v) - \Delta y(u) \\ &= \sum_{s=u}^{v-1} a^{\frac{-1}{\alpha}}(s)\left(a^{\frac{1}{\alpha}}(s)\Delta^2 y(s)\right) \\ &\geq S_1(v,u)\left[a(v)\left(\Delta^2 y(u)\right)^{\alpha}\right]^{\frac{1}{\alpha}}. \end{aligned}$$

Letting $u = \eta(t)$ and $v = \varsigma(t)$ in the last inequality, we have

$$-\Delta y(\eta(t)) \geq S_1(\varsigma(t),\eta(t))\left(a(\varsigma(t))\left(\Delta^2 y\left(\varsigma(t)\right)\right)^{\alpha}\right)^{\frac{1}{\alpha}}. \tag{26}$$

Combining (25) and (26) yields

$$y(h(t)) \geq (\eta(t) - h(t))S_1(y(t), \eta(t)) \left[a(\varsigma(t)) \left(\Delta^2 y(\varsigma(t))\right)^{\alpha}\right]^{\frac{1}{\alpha}}. \tag{27}$$

Using (27) in (25) gives

$$\Delta r(t) + c(t)p_2^{\frac{\gamma}{\beta}}(\sigma(t)) \left[(\eta(t) - h(t))S_1(\varsigma(t), \eta(t))\right]^{\frac{\alpha}{\beta}} r^{\frac{\gamma}{\alpha\beta}}(\varsigma(t)) \leq 0, \tag{28}$$

where $r(t) = a(t)\left(\Delta^2 y(t)\right)^{\alpha} > 0$. As in case (I), we see that there exists a positive solution of equation (21) which contradicts the fact that equation (21) is oscillatory.
In case (b), as in Lemma 2.4, we see that $x(t) \to 0$ as $t \to \infty$. This completes the proof. □

The following are immediate consequences of Theorem 2.6.

Corollary 2.7. *Let $\gamma = \alpha\beta$ and assume that conditions $[H_1]$-$[H_4]$ and (2) hold. Suppose that there exist positive real sequences $\{\eta(t)\}$ and $\{\varsigma(t)\}$ such that $h(t) \leq \eta(t) \leq \varsigma(t) \leq t-1$ for $t \geq t_0$. If*

$$\liminf_{t\to\infty} \sum_{s=h(t)}^{t-1} c(s)P_1^{\frac{\gamma}{\beta}}(\sigma(s))S_2^{\frac{\gamma}{\beta}}(h(s), t_0) > \frac{1}{e} \tag{29}$$

and

$$\liminf_{t\to\infty} \sum_{s=\varsigma(t)}^{t-1} c(s)P_2^{\frac{\gamma}{\beta}}(\sigma(s)) \left[(\eta(s) - h(s))S_1(\varsigma(s), \eta(s))\right]^{\frac{\gamma}{\beta}} > \frac{1}{e} \tag{30}$$

then every solution $x(t)$ of equation (1) either oscillates or satisfies
$\lim_{t\to\infty} x(t) = 0$.

Corollary 2.8. *Let $\gamma < \alpha\beta$ and assume that conditions $[H_1]$-$[H_4]$ and (2) hold. Suppose that there exist positive sequences $\{\eta t\}$ and $\{\varsigma(t)\}$ such that $h(t) \leq \eta(t) \leq \varsigma(t) \leq t-1$ for $t \geq t_0$. If for all sufficiently large $t_1 \geq t_0$ and for some $t_2 \geq t_1$*

$$\sum_{s=t_2}^{\infty} c(s)P_1^{\frac{\gamma}{\beta}}(\sigma(s))S_2^{\frac{\gamma}{\beta}}(h(s), t_0) = \infty \tag{31}$$

and

$$\sum_{s=t_0}^{\infty} c(s)p_s^{\frac{\gamma}{\beta}}(\sigma(s)) \left[(\eta(s) - h(s))S_1(\varsigma(s), \eta(s))\right]^{\frac{\gamma}{\beta}} = \infty \tag{32}$$

then every solution $x(t)$ of equation (1) either oscillates or satisfies
$\lim_{t\to\infty} x(t) = 0$.

The following are examples to illustrate the above results.

3 Examples

Example 3.1. *Consider the third order difference equation with a superlinear neutral term*

$$\Delta\left[\frac{1}{t^{\frac{1}{3}}}\left(\Delta^2 y(t)\right)^{\frac{1}{3}}\right]+\frac{2t}{3}x^3\left(\frac{t}{3}\right)=0, \qquad t\geq 2 \tag{33}$$

with $y(t)=x(t)+4\left(x^3\left(\frac{t}{2}\right)\right)$. *Here* $b(t)=4t$, $a(t)=\frac{1}{t^{\frac{1}{3}}}$, $c(t)=\frac{8t}{3}$, $\zeta(t)=\frac{t}{2}$, $\sigma(t)=\frac{t}{3}$, $\alpha=\frac{1}{3}$, $\beta=3$, $\gamma=3$ *and* $h(t)=\zeta^{-1}(\sigma(t))=\frac{2t}{3}$.
Then the conditions $[H_1]$-$[H_4]$ *and* (2) *hold.*

$$S_1(t,t_1)=S_1(t,2)=t-2$$

$$S_2(\zeta^{-1}(t),t_2)=S_2(2t,3)=2t-4$$
$$S_2\left(\zeta^{-1}(\zeta^{-1}(t)),t_2\right)=S_2(4t,3)=4t-8$$

and $P_2(t)=\frac{1}{8t}\left[1-\frac{l^{\frac{1}{3}-1}}{(16t)^{\frac{1}{3}}}\right]$, $P_1(t)=\frac{1}{8t}\left[1-\left(\frac{4t-8}{2t-4}\right)^{\frac{1}{3}}\frac{k^{\frac{1}{3}-1}}{(16t)^{\frac{1}{3}}}\right]$. *Thus conditions* (16) *and* (17) *hold. Hence by Theorem 2.5 any solution* $x(t)$ *of equation* (33) *is either oscillatory or satisfies* $\lim_{t\to\infty}x(t)=0$.

Example 3.2. *Consider the third order difference equation with a linear neutral term*

$$\Delta\left[\frac{1}{t^{\frac{1}{5}}}\left(\Delta^2 y(t)\right)^{\frac{1}{5}}\right]+(1+t^2)x^{\frac{1}{5}}\left(\frac{t}{8}\right)=0, \qquad t\geq 12 \tag{34}$$

with $y(t)=x(t)+16x\left(\frac{t}{2}\right)$. *Here* $a(t)=\frac{1}{t^{\frac{1}{5}}}$, $b(t)=8$, $c(t)=(1+t^2)$, $\zeta(t)=\frac{t}{2}$, $\sigma(t)=\frac{t}{8}$, $\alpha=\frac{1}{5}$, $\beta=1$, $\gamma=\frac{1}{5}$, $h(t)=\frac{t}{4}$.
Then the conditions $[H_1]$-$[H_4]$ *and* (2) *hold.*

$$S_1(t,2)=t-2$$

$$S_2(h(t),2)=S_2\left(\frac{t}{4},2\right)=\frac{t-12}{4}$$

$$\eta(t)=\frac{t}{3},\varsigma(t)=\frac{t}{2}$$

$$S_1(\varsigma(t),\eta(t))=S_1\left(\frac{t}{2},\frac{t}{3}\right)=\frac{t}{6}.$$

Thus conditions (31) *and* (32) *hold. Hence by Corollary 2.8, any solution* $x(t)$ *of equation* (34) *either oscillates or satisfies* $\lim_{t\to\infty}x(t)=0$.

4 Concluding Remarks

This paper is presented in the form which is essentially new. The results obtained are different from many known theorems reported in the literature. By comparision method, the oscillatory and asymptotic behavior of every solution of equation (1.1) are discussed in Theorems 2.5 and 2.6. Examples reveal the illustration of the proved results.

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References

[1] Agarwal R.P, Bohner M, Grace S.R and O'Regen D. *Discrete Oscillation Theory*, Hindawi Publishing corporation. New York, 2005.

[2] R.P. Agarwal, *Difference Equations and Inqualities, Theory, Methods and Applications*, Second Edition, Revised and Expanded, New York, Marcel Dekker, 2000.

[3] R. P. Agarwal, P. J. Y. wong, *Advanced Topics in Difference Equations*, Kluwer Academic Publishers, Dordrecht,1997.

[4] R. P. Agarwal, Said R. Grace, Donal O'Regan, *Oscillation Theory for Difference and Functional Differential Equations*, Kluwer Academic Publishers, Dordrecht, 2000.

[5] R. P. Agarwal, Said R. Grace, Donal O'Regan, *On the oscillation of higher order difference equations*, Soochow Journal of Mathematics, **Vol. 31**(2005), No. 2, pp. 245-259.

[6] S. R. Grace, J. R. Graef and E. Tunc, *Oscillatory behavior of third order nonlinear differential equations with a nonlinear nonpositive neutral term*, J.Taibah Univ. Sci. **13**(2019), pp. 704-710.

[7] J. R. Graef, S. R. Grace and E. Tunc, *Oscillatory behavior of even-order nonlinear differential equations with a sublinear neutral term*, Opuscula Math., **Vol. 39**(2019), No.1, pp. 39â47.

[8] J. R. Graef, E. Tunc and S. R. Grace, *Oscillatory and asymptotic behavior of a third order nonlinear neutral differential equations*, Opuscula Math., **37**(6), (2017), pp. 839-852.

[9] I. Gyori, G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, Clarendon Press, Oxford, 1991.

[10] S. Jaikumar, K. Alagesan and G. Ayyappan, *Oscillation of nonlinear third order delay difference equations with unbounded neutral coefficients*, J. Inf. Comput. Sci. **9**(2019), pp. 902-910.

[11] S. Kaleeswari, *On the oscillation of higher order nonlinear neutral difference equations*, Advances in Difference Equations, **2019**(1), (2019) pp. 1-10.

[12] S. Kaleeswari, *Oscillatory and asymptotic behavior of third order mixed type neutral difference equations*, Journal of Physics: Conference Series **1543** (1), 012005(2020).

[13] S. Kaleeswari, *Oscillation Criteria For Mixed Neutral Difference Equations*, Asian Journal of Mathematics and Computer Research, textbf25(6), 2018 pp. 331-339(2018)

[14] S. Kaleeswari, B. Selvaraj, *On the oscillation of certain odd order nonlinear neutral difference equations*, Applied Sciences **18**, (2016), pp. 50-59.

[15] S. Kaleeswari, B. Selvaraj and M. Thiyagarajan, *A New Creation of Mask From Difference Operator to Image Analysis*, Journal of Theoretical and Applied Information Technology, **Vol. 69**(1)(2014), pp.211-218.

[16] S. Kaleeswari, B. Selvaraj and M. Thiyagarajan, *Removing Noise Through a Nonlinear Difference Operator*, International Journal of Applied Engineering Research, **Vol. 9**(21)(2014), pp.5100-5105.

[17] S. Kaleeswar, B. Selvaraj and M. Thiyagarajan, *An Application of Certain Third Order Difference Equation in Image Enhancement*, Asian Journal of Information Technology, **15** (23), (2016) pp. 4945-4954

[18] W. G. Kelley, A. C. Peterson, *Difference Equations an Introduction with Applications*, Academic Press, Boston, 1991.

[19] Ch. G. Philos, *On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delays*, Arch. Math. (Basel), **36**, (1981), pp. 168-178.

[20] B. Selvaraj and S. Kaleeswari, *Oscillation of Solutions of Certain Nonlinear Difference Equations*, Progress in Nonlinear Dynamics and Chaos, **Vol. 1**, 2013, 34-38.

[21] B. Selvaraj and S. Kaleeswari, *Oscillation Theorems for Certain Fourth Order Non-linear Difference Equations*, International Journal of Mathematics Research, **Volume 5**, Number 3 (2013), pp. 299-312.

[22] B. Selvaraj and S. Kaleeswari, *Oscillation of Solutions of second Order Nonlinear Difference Equations*, Bulletin of Pure and Applied Sciences, **Volume 32 E** (Math and Stat.) Issue (No.1), 2013, P. 107-117.

[23] E. Thandapani and B. Selvaraj, *Existence and Asymptotic Behavior of Non Oscillatory Solutions of Certain Nonlinear Difference Equations*, Far East Journal of Mathematical Sciences (FJMS), **14**(1) (2004), 9 - 25.

[24] E. Tunc, S. R. Grace, *Oscillatory behavior of solutions to third order nonlinear differential equations with a superlinear neutral term*, Electronic Journal of Differential equations, **32**, (2020), pp. 1-11.

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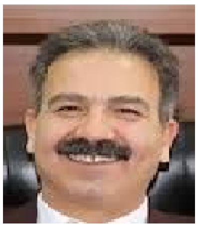

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