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27th October 2021

Jointly Organized by

Department of Biological Science, Physical Science and Computational Science

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ABOUT THE INSTITUTION

A nations's growth is in proportion to education and intelligence spread among the masses. Having this idealistic vision, two great philanthropists late. S.P. Nallamuthu Gounder and Late. Arutchelver Padmabhushan Dr.N.Mahalingam formed an organization called Pollachi Kalvi Kazhagam, which started NGM College in 1957, to impart holistic education with an objective to cater to the higher educational needs of those who wish to aspire for excellence in knowledge and values. The College has achieved greater academic distinctions with the introduction of autonomous system from the academic year 1987-88. The college has been Re-Accredited by NAAC and it is ISO 9001 : 2015 Certified Institution. The total student strength is around 6000. Having celebrated its Diamond Jubilee in 2017, the college has blossomed into a premier Post-Graduate and Research Institution, offering 26 UG, 12 PG, 13 M.Phil and 10 Ph.D Programmes, apart from Diploma and Certificate Courses. The college has been ranked within Top 100 (72nd Rank) in India by NIRF 2021.

ABOUT CONFERENCE

The International conference on “Emerging Trends in Science and Technology (ETIST-2021)” is being jointly organized by Departments of Biological Science, Physical Science and Computational Science - Nallamuthu Gounder Mahalingam College, Pollachi along with ISTE, CSI, IETE, IEE & RIYASA LABS on 27th OCT 2021. The Conference will provide common platform for faculties, research scholars, industrialists to exchange and discus the innovative ideas and will promote to work in interdisciplinary mode.

Solution of Linear Fuzzy VolterraIntegro-Differential Equations using Generalized Differentiability

***S Indrakumar[1], K Kanagarajan[2], R.Santhi[3]**

ABSTRACT: Fuzzy VolterraIntegro-differential equations are well adapted to describe physical phenomena, when vagueness and uncertainty exists. In the perspective of growing applications of fuzzy VolterraIntegro-differential equations, this article discuss the solution of fuzzy VolterraIntegro-differential equations under generalized differentiability using Runge-Kutta method of fourth order. The proposed method was validated by numerical examples. The comparison of exact and approximate solutions was tabulated with the corresponding errors. The solution was also visualized.

KEYWORDS: Fuzzy Number, Generalized Differentiability, Fuzzy VolterraIntegro-Differential Equations, Fourth Order Runge-Kutta Method.

1. INTRODUCTION

Fuzzy set theory initiated by Zadeh [44] had widen its domain by its fusion with proven assertion and well-established theories and had strengthened its ideas in dealing with data by which it is now an independent branch of Applied Mathematics. Dubois and Prade [11] established fuzzy calculus using extension principle and fuzzy integration. Riemann integral oriented fuzzy calculus was examined by Goetschal and Voxman [15]. Lebesque model was preffered by Kaleva [30] for integration of fuzzy functions. Arbitrary Kernal was used in fuzzy integral equations by Lakshmikantham and Mohapatra [20]. The theories under fuzzy integral equations were under investigation by many researchers [6, 13, 28, 31, 33, 35, 38]. FVIDE with fuzzy set valued mappings combined by existence and uniqueness theorems was put forth by Hajighasemiet. al. [25]. The concept of handling Numerical methods in the solution of fuzzy integral equations along with arbitrary kernals was studied by Friedman et.al. [23]. The solution of Volterraintegro-differential equations (VIDE) of second kind was sought utilizing Runge-kutta methods was studied by Lubich [21]. Allviranlooet. al. [3] founded the solution of fuzzy integro-differential equations (FIDE) under strongly generalized H-differentiability by considering convex combination of 0-cut and 1-cut solutions. Alikhani and Bahrami [36] discussed the global solutions of FIDE by the method of upper and lower

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solutions under generalized differentiability. Matinfaret. al. [24] studied variational iteration technique for establishing numerical solution of FVIDE. In this work, we use Runge-Kutta method of order four for solving linear FVIDE of the second kind under generalized differentiability.

This paper is frame worked as follows. The preliminary ideas and basic concepts were discussed in section 2. Linear FVIDE under generalized differentiability was studied in section 3. Section 4 analyses the extraction of numerical solution of FVIDE with Runge-Kutta method of order four under generalized differentiability. Finally in section 5, the proposed method is illustrated with numerical examples.classic decision making problem may not be satisfied, when the decision situation involves both fuzzy and crisp data. The classical decision making methods cannot handle such problems effectively, because they are only suitable for dealing with problems in which all performances of the criteria are represented by crisp numbers. The application of the fuzzy set theory in the field of decision making is justified when the intended goals or their attainment cannot be defined or judged crisply but only as fuzzy sets.

2. PRELIMINARIES

In this section, some basic notations of fuzzy integral equations and the concept of generalized differentiability were provided.

DEFINITION 2.1: A fuzzy number is a fuzzy set $u : R \rightarrow R_F = [0,1]$ which satisfies

(i) u is normal, i.e., there exist $t_0 \in R$ such that $u(t_0) = 1,$

(ii) u is fuzzy convex, i.e., there exist

$u(\beta t_1 + (1-\beta t_2)) \geq \{u(t_1), u(t_2)\}$ for any $t_1, t_2 \in R, \quad \beta \in [0,1]$,

(iii) u is upper semi-continuous,

(iv) $[u]^0 = cl\ \{t \in R | u(t) > 0\}$ is compact

The space of fuzzy numbers is denoted by R_F . Obviously $R \subset R_F$.

DEFINITION 2.2: A fuzzy number u is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r)$ and $\overline{u}(r)$ $0 \leq r \leq 1$, which satisfy the following conditions are

(i) $\underline{u}(r)$ is a bounded monotonically increasing, left continuous function on $(0,1]$ and right continuous at 0,

(ii) $\overline{u}(r)$ is a bounded monotonically increasing, left continuous function on $(0,1]$ and right continuous at 0,

(iii) $\underline{u}(r) \leq \overline{u}(r), 0 \leq r \leq 1.$

DEFINITION 2.3: If $u,v \in R_F$ and if there exist a fuzzy number $w \in R_F$ such that $u = v + w,$ then w is called the H-difference of u,v and it is denoted by $u \ominus v$.

Now, " $\ominus$ " stands for the H-difference and let us remark that $u \ominus v \neq u + (-1)v$. Usually we denote $u + (-1)v$ by $u - v$ stands for the H-difference.

If $u,v \in R_F$, the distance between u and v is defined by

$D(u,v) = \sup_{0 \le r \le 1} \max\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\}$. (R_F, D) is a complete metric space and the following properties are

$$D(u+w, v+w) = D(u,v), \quad \forall u,v,w \in R_F,$$

$$D(k \odot u, k \odot v) = |k| D(u,v), \qquad \forall k \in R, u,v \in R_F,$$

$$D(u+w, v+e) \le D(u,w) + D(v,e), \qquad \forall u,v,w,e \in R_F,$$

In this paper, for the integral concept, we will use the fuzzy Riemann integral and also if $F:[a,b] \to R_F$, be integrable fuzzy function and denote $[F(t;r)] = [\underline{F}(t;r), \overline{F}(t;r)]$. Then the boundary functions $\underline{F}(t;r)$ and $\overline{F}(t;r)$ are integrable and $\left[\int_a^b F(t;r)dt\right] = \left[\int_a^b \underline{F}(t;r)dt, \int_a^b \overline{F}(t;r)dt\right], r \in [0,1]$.

DEFINITION 2.4: A function $F:(a,b) \to R_F$, is called H-differentiable at $t_0 \in (a,b)$ if for $h > 0$ sufficiently small there exist the H-differences $F(t_0+h) \ominus F(t_0), F(t_0) \ominus F(t_0-h)$ and an element $F'(t_0) \in R_F$ such that

$$\lim_{h \to 0^+} D\left(\frac{F(t_0+h) \ominus F(t_0)}{h}, F'(t_0)\right) = \lim_{h \to 0^+} D\left(\frac{F(t_0) \ominus F(t_0-h)}{h}, F'(t_0)\right) = 0.$$

Then $F'(t_0)$ is called the fuzzy derivative of F at t_0.

DEFINITION 2.5: Let $F:(a,b) \to R_F$ and $t_0 \in (a,b)$. We say F is generalized differentiable at t_0, if there exist an element $F'(t_0) \in R_F$ such that:

(1) for all $h > 0$ sufficiently small, there exist $F(t_0+h) \ominus F(t_0), F(t_0) \ominus F(t_0-h)$ and the limits (in the metric D)

$$\lim_{h \to 0} \frac{F(t_0+h) \ominus F(t_0)}{h} = \lim_{h \to 0} \frac{F(t_0) \ominus F(t_0-h)}{h} = F'(t_0) \text{(or)}$$

(2) for all $h > 0$ sufficiently small, thereexist $F(t_0) \ominus F(t_0-h), F(t_0+h) \ominus F(t_0)$, and the limits (in the metric D)

$$\lim_{h\to 0}\frac{F(t_0)\ominus F(t_0+h)}{-h}=\lim_{h\to 0}\frac{F(t_0-h)\ominus F(t_0)}{-h}=F'(t_0),$$

(*h* and –*h* at denominator mean $\frac{1}{h}$ and $-\frac{1}{h}$ respectively).

DEFINITION 2.6: Let $F:(a,b)\to R_F$, be a fuzzy function and $[F(t)]^r=[\underline{F}(t;r),\overline{F}(t;r)]$ for each $r\in[0,1]$.

(i) If F is (1)-differentiable then $\underline{F}_r$ and $\overline{F}_r$ are differentiable functions and $[D_1F(t)]^r=[\underline{F}'(t;r),\overline{F}'(t;r)]$.

(ii) If F is (2)-differentiable then $\underline{F}_r$ and $\overline{F}_r$ are differentiable functions and $[D_2F(t)]^r=[\overline{F}'(t;r),\underline{F}'(t;r)]$.

DEFINITION 2.7: Let $F:[a,b]\to R_F$, for each partition $P=\{t_0,t_1,\ldots,t_n\}$ of $[a,b]$ and for arbitrary $\xi_i\in[t_{i-1},t_i], 1\le i\le n$ suppose

$$R_P=\sum_{i=1}^{n}F(\xi)(t_i-t_{i-1}),\Delta:=\max\{|t_i-t_{i-1}|,i=1,\ldots,n\}$$

The definite integral of $F(t)$ over $[a,b]$ is

$$\int_a^b F(t)dt=\lim_{\Delta\to 0}R_P,$$

provide that this limit exists in the metric *D* [14,15]. If the fuzzy function $F(t)$. Is continuous in the metric *D*, its definite integral exists [15]. Also,

$$\underline{\left(\int_a^b F(t;r)dt\right)}=\int_a^b \underline{F}(t;r)dt,\ \overline{\left(\int_a^b F(t;r)dt\right)}=\int_a^b \overline{F}(t;r)dt.$$

3. FVIDE UNDER GENERALIZED DIFFERENTIABILITY

In this section, the concept of generalized differentiability was employed to convert fuzzy Volterraintegro-differential equations in to a system of Volterraintegro-differential equations.

Let us consider the linear FVIDE of the form

$$y'(t)=f(t,y(t))+\lambda\int_a^t g(t,s,y(s))ds,\qquad t\in J,$$
$$y(t_0)=y_0, \tag{1}$$

where $\lambda>0$, $f(t,y(t))$ is continuous fuzzy function $(f\in C(J\times R_F,R_F))$ on the interval $J(J\in[a,b])$, $g(t,s,y(s))$ is an arbitrary kernel function $(g\in C(I\times R_F,R_F))$ over $I=\{(t,s)\in J\times J: a\le s\le t\le b\}$ and $x_0\in R_F$, The parametric form of $f(t, y(t)),\ y'(t)$ and $g(t,s,y(s))$ are given as $(f_1(t, \underline{y}(t;r), \overline{y}(t;r)),$ $f_2(t, \underline{y}(t;r), \overline{y}(t;r)))$, $(\underline{y}'(t;r), \overline{y}'(t;r))$ and $(g_1(t,s,\underline{y}(s;r),\overline{y}(s;r)), g_2(t,s,\underline{y}(s;r),\overline{y}(s;r)))$ (we assume that the function g takes the form $g(t,s,y(s)) = K(t,s)G(y(s))$ so based on this FVIDE Eq (1) can be written in a new discretized form as $y'(t) = f(t, y(t)) + \int_a^t K(t,s)G(y(s))ds$ in which the r-cut representation form of $G(y(s))$ should be of the form $[G(y(s))]^r = [G_1(\underline{y}(s;r),\overline{y}(s;r)), G_2(\underline{y}(s;r),\overline{y}(s;r))]$) $0 \le r \le 1$, and $a \le t \le b$.

DEFINITION 3.1: Let $y: J \to R_F$, such that $D_1 y$ and $D_2 y$ exists. If y and $D_1 y$ satisfy FVIDE (Eq (1)) we say that *y* is a (1)-differentiable of FVIDE (Eq (1)). Similarly, y and $D_1 y$ satisfy FVIDE (Eq (1)) we say that *y* is a (2)-differentiable of FVIDE (Eq (1)).

(1) By using the generalized differentiability concept we convert the FVIDE into the system of VIDE as follows.

If $x(t)$ is (1)-differentiable, then $D_1 y(t;r) = [\underline{y}'(t;r), \overline{y}'(t;r)]$ and Eq (1) translates into the following system of VIDE

$$\begin{aligned}
\underline{y}'(t;r) &= f_1(t,\underline{y}(t;r),\overline{y}(t;r)) + \lambda\int_a^t g_1(t,s,\underline{y}(s;r),\overline{y}(s;r))ds \\
&= U_1(t,\underline{y}(t),\overline{y}(t)) + \lambda\int_a^t V_1(t,s,\underline{y}(s),\overline{y}(s))ds, \\
\underline{y}(t_0) &= \underline{y}_0, \\
\overline{y}'(t;r) &= f_2(t,\underline{y}(t;r),\overline{y}(t;r)) + \lambda\int_a^t g_2(t,s,\underline{y}(s;r),\overline{y}(s;r))ds \\
&= U_2(t,\underline{y}(t),\overline{x}(t)) + \lambda\int_a^t V_2(t,s,\underline{y}(s),\overline{y}(s))ds, \\
\overline{y}(t_0) &= \overline{y}_0,
\end{aligned} \tag{2}$$

where

$$V_1(t,s,\underline{y}(s),\overline{y}(s)) = \begin{cases} K(t,s)G_1(\underline{y}(s),\overline{y}(s)), & K(t,s) \ge 0, \\ K(t,s)G_2(\underline{y}(s),\overline{y}(s)), & K(t,s) < 0, \end{cases}$$

and

$$V_2(t,s,\underline{y}(s),\overline{y}(s)) = \begin{cases} K(t,s)G_2(\underline{y}(s),\overline{y}(s)), & K(t,s) \ge 0, \\ K(t,s)G_1(\underline{y}(s),\overline{y}(s)), & K(t,s) < 0. \end{cases}$$

(1) If $x(t)$ is (2)-differentiable, then $D_2 y(t;r) = [\overline{y}'(t;r), \underline{y}'(t;r)]$ and Eq (1) translates into the following system of VIDE

$$
\begin{aligned}
\underline{y}'(t;r) &= f_2(t, \underline{y}(t;r), \overline{y}(t;r)) + \lambda \int_a^t g_2(t,s,\underline{y}(s;r), \overline{y}(s;r))ds \\
&= U_2(t, \underline{y}(t), \overline{y}(t)) + \lambda \int_a^t V_2(t,s,\underline{y}(s), \overline{y}(s))ds, \\
\underline{y}(t_0) &= \underline{y}_0, \\
\overline{y}'(t;r) &= f_1(t, \underline{y}(t;r), \overline{y}(t;r)) + \lambda \int_a^t g_1(t,s,\underline{y}(s;r), \overline{y}(s;r))ds \\
&= U_1(t, \underline{y}(t), \overline{y}(t)) + \lambda \int_a^t V_1(t,s,\underline{y}(s), \overline{y}(s))ds, \\
\overline{y}(t_0) &= \overline{y}_0,
\end{aligned} \tag{3}
$$

where

$$
V_1(t,s,\underline{y}(s),\overline{y}(s)) = \begin{cases} K(t,s)G_2(\underline{y}(s),\overline{y}(s)), & K(t,s) \geq 0, \\ K(t,s)G_1(\underline{y}(s),\overline{y}(s)), & K(t,s) < 0, \end{cases}
$$

and

$$
V_2(t,s,\underline{y}(s),\overline{y}(s)) = \begin{cases} K(t,s)G_1(\underline{y}(s),\overline{y}(s)), & K(t,s) \geq 0, \\ K(t,s)G_2(\underline{y}(s),\overline{y}(s)), & K(t,s) < 0. \end{cases}
$$

for each $a \leq t \leq b$.

THEOREM 3.2: Consider the FVIDE Eq(1) where $f : J \times R_F \to R_F$, $g : J \times J \times R_F \to R_F$ are such that

(i) $[f(t,y(t))]^r = [\underline{f}(t,\underline{y}(t;r),\overline{y}(t;r)), \overline{f}(t,\underline{y}(t;r),\overline{y}(t;r))]$,

$[g(t,s,y(s))]^r = [\underline{g}(t,s,\underline{y}(s;r),\overline{y}(s;r)), \overline{g}(t,s,\underline{y}(s;r),\overline{y}(s;r))]$,

(ii) $\underline{f}(t:r), \overline{f}(t:r)$ and $\underline{g}(t;r), \overline{g}(t;r)$ are equicontinuous functions,

(iii) there exist $L_1, L_2, L_3, L_4 > 0$ are real constants

$$
\begin{aligned}
&|\underline{f}(t,\underline{u}(t;r),\overline{u}(t;r)) - \underline{f}(t,\underline{v}(t;r),\overline{v}(t;r))| \leq L_1 \max\{|\underline{u}(t;r)-\underline{v}(t;r)|, |\overline{u}(t;r)-\overline{v}(t;r)|\}, \\
&|\overline{f}(t,\underline{u}(t;r),\overline{u}(t;r)) - \overline{f}(t,\underline{v}(t;r),\overline{v}(t;r))| \leq L_2 \max\{|\underline{u}(t;r)-\underline{v}(t;r)|, |\overline{u}(t;r)-\overline{v}(t;r)|\}, \\
&|\underline{g}(t,s,\underline{u}(s;r),\overline{u}(s;r)) - \underline{g}(t,s,\underline{v}(s;r),\overline{v}(s;r)) \leq L_3 \max\{|\underline{u}(s;r)-\underline{v}(s;r)|, |\overline{u}(s;r)-\overline{v}(s;r)|\}, \\
&|\overline{g}(t,s,\underline{u}(s;r),\overline{u}(s;r)) - \overline{g}(t,s,\underline{v}(s;r),\overline{v}(s;r)) \leq L_4 \max\{|\underline{u}(s;r)-\underline{v}(s;r)|, |\overline{u}(s;r)-\overline{v}(s;r)|\},
\end{aligned}
$$

for each $t, s, \in J$, $r \in [0,1]$ and $u(t), v(t), u(s), v(s) \in R_F$. Then for (1)-differentiability, the FVIDE Eq (1) and the system of VIDE Eq (2) are equivalent and in (2)-differentiability, the FVIDE Eq (1) and the system of VIDE Eq (3) are equivalent.

4. RUNGE-KUTTA METHOD FOR FVIDE UNDER GENERALIZED DIFFERENTIABILITY

The method of solving FVIDE by Runge-Kutta method of order four under generalized differentiability was developed in this section in detail.

LEMMA 4.1: Eq (1) is equivalent to one of the following fuzzy integral equations

$$y(t) = y_0 + \left(\int_a^t f(s, y(s))ds + \lambda \int_a^t \left(\int_a^s g(s,u,y(u))du,\right) ds\right), \quad t \in J,$$

or

$$y_0 = y(t) + (-1) \odot \left(\int_a^t f\big(s, y(s)\big)ds + \lambda \int_a^t \left(\int_a^s g(s,u,y(u))du\right)ds\right), t\epsilon J,$$

depending on the strongly differentiability considered (1)-differentiability or (2)-differentiability, respectively.

LEMMA 4.2: The FVIDE Eq (1) considered under generalized differentiability has locally two solutions, and the successive iterations

$$y(0) = y_0, \qquad or \qquad y_{n+1}(t) = y_0 + \left(\int_a^t f(s, y_n(s))ds + \lambda \int_a^t \left(\int_a^s g(s,u,y_n(u))du,\right) ds\right), \quad \text{or}$$

$$y(t) = y_0, \quad y_{n+1}(t) = y_0 \ominus (-1) \left(\int_a^t f\big(s, y_n(s)\big)ds + \lambda \int_a^t \left(\int_a^s g(s,u,y_n(u))du\right)ds\right),$$

converge to the (1)-differentiability or (2)-differentiability, respectively.

The Runge-Kutta method for FVIDE under (1)-differentiability is given by the formula

$$\underline{y}_{1_{n+1}}(r) = \left\{ \underline{y}_{1_n}(r) + h\sum_{i=1}^m b_i\left[F_1\left(t_n + c_i h, u\right) + v + w\right] \middle| \begin{array}{l} u \in \left[\underline{U}_{1_{n,i}}(r), \overline{U}_{1_{n,i}}(r)\right], \\ v \in \left[\underline{z}_{1_n}\left(t_n + c_i h; r\right), \overline{z}_{1_n}\left(t_n + c_i h; r\right)\right], \\ w \in \left[\underline{Z}_{1_{n,i}}(r), \overline{Z}_{1_{n,i}}(r)\right] \end{array} \right\}, \tag{4}$$

$$\overline{y}_{1_{n+1}}(r) = \left\{ \overline{y}_{1_n}(r) + h\sum_{i=1}^m b_i\left[F_2\left(t_n + c_i h, u\right) + v + w\right] \middle| \begin{array}{l} u \in \left[\underline{U}_{1_{n,i}}(r), \overline{U}_{1_{n,i}}(r)\right], \\ v \in \left[\underline{z}_{1_n}\left(t_n + c_i h; r\right), \overline{z}_{1_n}\left(t_n + c_i h; r\right)\right], \\ w \in \left[\underline{Z}_{1_{n,i}}(r), \overline{Z}_{1_{n,i}}(r)\right] \end{array} \right\},$$

where

$$\underline{U_1}_{n,i}(r)=\left\{\underline{y_1}_n(r)+h\sum_{j=1}^{m}a_{ij}\left[F_1\left(t_n+c_jh,u\right)+v+w\right]\middle|\begin{array}{l}u\in\left[\underline{U_1}_{n,j}(r),\overline{U}_{1n,j}(r)\right],\\ v\in\left[\underline{z_1}_n\left(t_n+c_jh;r\right),\overline{z}_{1n}\left(t_n+c_jh;r\right)\right],\\ w\in\left[\underline{Z_1}_{n,j}(r),\overline{Z}_{1n,j}(r)\right]\end{array}\right\},$$

$$\overline{U}_{1n,i}(r)=\left\{\overline{y}_{1n}(r)+h\sum_{j=1}^{m}a_{ij}\left[F_2\left(t_n+c_jh,u\right)+v+w\right]\middle|\begin{array}{l}u\in\left[\underline{U_1}_{n,j}(r),\overline{U}_{1n,j}(r)\right],\\ v\in\left[\underline{z_1}_n\left(t_n+c_jh;r\right),\overline{z}_{1n}\left(t_n+c_jh;r\right)\right],\\ w\in\left[\underline{Z_1}_{n,j}(r),\overline{Z}_{1n,j}(r)\right]\end{array}\right\},$$

$$\underline{Z_1}_{n,i}(r)=\left\{h\sum_{j=1}^{m}e_{ij}V_1\left(t_n+d_{ij}h,s_n+c_jh,u\right)\middle|u\in[\underline{U_1}_{n,j}(r),\overline{U}_{1n,j}(r)]\right\},$$

$$\overline{Z}_{1n,i}(r)=\left\{h\sum_{j=1}^{m}e_{ij}V_2\left(t_n+d_{ij}h,s_n+c_jh,u\right)\middle|u\in[\underline{U_1}_{n,j}(r),\overline{U}_{1n,j}(r)]\right\},$$

$$\underline{z_1}_n(t_n+c_jh;r)=\left\{h\sum_{j=0}^{k-1}\sum_{i=1}^{m}b_iV_1\left(y_n+c_jh,y_j+c_ih,u\right)\middle|u\in[\underline{U_1}_{j,i}(r),\overline{U}_{1j,i}(r)]\right\},$$

$$\overline{z}_{1n}(t_n+c_jh;r)=\left\{h\sum_{j=0}^{k-1}\sum_{i=1}^{m}b_iV_2\left(y_n+c_jh,y_j+c_ih,u\right)\middle|u\in[\underline{U_1}_{j,i}(r),\overline{U}_{1j,i}(r)]\right\}$$

$$(\forall\ i=1,\dots,m).$$

Similarly, the Runge-Kutta method for FVIDE under (2)-differentiability is given by the formula

$$\underline{y_2}_{n+1}(r)=\left\{\underline{y_2}_n(r)+h\sum_{i=1}^{m}b_i\left[F_2\left(t_n+c_ih,u\right)+v+w\right]\middle|\begin{array}{l}u\in\left[\underline{U_2}_{n,i}(r),\overline{U}_{2n,i}(r)\right],\\ v\in\left[\underline{z_2}_n\left(t_n+c_ih;r\right),\overline{z}_{2n}\left(t_n+c_ih;r\right)\right],\\ w\in\left[\underline{Z_2}_{n,i}(r),\overline{Z}_{2n,i}(r)\right]\end{array}\right\},\quad(5)$$

$$\overline{y}_{2n+1}(r)=\left\{\overline{y}_{2n}(r)+h\sum_{i=1}^{m}b_i\left[F_1\left(t_n+c_ih,u\right)+v+w\right]\middle|\begin{array}{l}u\in\left[\underline{U_2}_{n,i}(r),\overline{U}_{2n,i}(r)\right],\\ v\in\left[\underline{z_2}_n\left(t_n+c_ih;r\right),\overline{z}_{2n}\left(t_n+c_ih;r\right)\right],\\ w\in\left[\underline{Z_2}_{n,i}(r),\overline{Z}_{2n,i}(r)\right]\end{array}\right\},$$

where

$$\underline{U_2}_{n,i}(r)=\left\{\underline{y_2}_n(r)+h\sum_{j=1}^{m}a_{ij}\left[F_2\left(t_n+c_jh,u\right)+v+w\right]\middle|\begin{array}{l}u\in\left[\underline{U_2}_{n,j}(r),\overline{U}_{2n,j}(r)\right],\\ v\in\left[\underline{z_2}_n\left(t_n+c_jh;r\right),\overline{z}_{2n}\left(t_n+c_jh;r\right)\right],\\ w\in\left[\underline{Z_2}_{n,j}(r),\overline{Z}_{2n,j}(r)\right]\end{array}\right\},$$

$$\overline{U}_{2n,i}(r)=\left\{\overline{y}_{2n}(r)+h\sum_{j=1}^{m}a_{ij}\left[F_1\left(t_n+c_jh,u\right)+v+w\right]\left|\begin{array}{l}u\in\left[\underline{U}_{2n,j}(r),\overline{U}_{2n,j}(r)\right],\\ v\in\left[\underline{z}_{2n}\left(t_n+c_jh;r\right),\overline{z}_{2n}\left(t_n+c_jh;r\right)\right],\\ w\in\left[\underline{Z}_{2n,j}(r),\overline{Z}_{2n,j}(r)\right]\end{array}\right.\right\},$$

$$\underline{Z}_{2n,i}(r)=\left\{h\sum_{j=1}^{m}e_{ij}V_2\left(t_n+d_{ij}h,s_n+c_jh,u\right)\middle|u\in\left[\underline{U}_{2n,j}(r),\overline{U}_{2n,j}(r)\right]\right\},$$

$$\overline{Z}_{2n,i}(r)=\left\{h\sum_{j=1}^{m}e_{ij}V_1\left(t_n+d_{ij}h,s_n+c_jh,u\right)\middle|u\in\left[\underline{U}_{2n,j}(r),\overline{U}_{2n,j}(r)\right]\right\},$$

$$\underline{z}_{2n}(t_n+c_jh;r)=\left\{h\sum_{j=0}^{k-1}\sum_{i=1}^{m}b_iV_2\left(x_n+c_jh,x_j+c_ih,u\right)\middle|u\in\left[\underline{U}_{2j,i}(r),\overline{U}_{2j,i}(r)\right]\right\}$$

$$\overline{z}_{2n}(t_n+c_jh;r)=\left\{h\sum_{j=0}^{k-1}\sum_{i=1}^{m}b_iV_1\left(x_n+c_jh,x_j+c_ih,u\right)\middle|u\in\left[\underline{U}_{2j,i}(r),\overline{U}_{2j,i}(r)\right]\right\},$$

$$(\forall\, i=1,\ldots,m).$$

We assume $c_i=\sum_{j=1}^{m}a_{ij}\,(i=1,\ldots,m)$ and $d_{ij}\geq c_j$ whenever $e_{ij}\neq 0$ to ensure that the argument V_1,V_2 inEq (4) and (5) are in the domain V_1,V_2. The constants c_i,b_i,a_{ij},e_{ij} in explicit Runge-Kutta method of order four $m=4$ for FVIDE which satisfy $a_{ij}=e_{ij}$ are given by

$$c_1=0,c_2=\frac{1}{2},c_3=\frac{1}{2},c_4=1,a_{11}=a_{22}=a_{33}=a_{41}=a_{42}=a_{44}=0,a_{21}=\frac{1}{2},a_{31}=\frac{4}{9},a_{32}=\frac{1}{18},a_{43}=1,b_1=\frac{1}{6},b_2=\frac{1}{3},b_3=\frac{1}{3},b_4=\frac{1}{6}$$

and we have Runge-Kutta method of order four for FVIDE under generalized differentiability based on the approximation of $\underline{y}_1(t;r),\overline{y}_1(t;r),\underline{y}_2(t;r),\overline{y}_2(t;r)$ and Eq (4) and (5) as follows,

for the (1)-differentiability

$$\underline{Z}_{1n,2}(r)=\left\{\frac{h}{2}\left[V_1\left(t_n+\frac{h}{2},s_n+\frac{h}{2},u\right)\right]\middle|u\in\left[\underline{U}_{1n,1}(r),\overline{U}_{1n,1}(r)\right]\right\},$$

$$\overline{Z}_{1n,2}(r)=\left\{\frac{h}{2}\left[V_2\left(t_n+\frac{h}{2},s_n+\frac{h}{2},u\right)\right]\middle|u\in\left[\underline{U}_{1n,1}(r),\overline{U}_{1n,1}(r)\right]\right\},$$

$$\underline{U}_{1\,n,3}(r)=\left\{\underline{x}_n+h\begin{bmatrix}\frac{4}{9}\left[F_1(t_n,u_1)+v_1+w_1\right]+\\ \frac{1}{18}\left[F_1(t_n+\frac{h}{2},u_2)+v_2+w_2\right]\end{bmatrix}\middle|\begin{matrix}u_1\in\left[\underline{U}_{1\,n,1}(r),\overline{U}_{1\,n,1}(r)\right],u_2\in\left[\underline{U}_{1\,n,2}(r),\overline{U}_{1\,n,2}(r)\right],\\ w_1\in\left[\underline{Z}_{1\,n,1}(r),\overline{Z}_{1\,n,1}(r)\right],w_2\in\left[\underline{Z}_{1\,n,2}(r),\overline{Z}_{1\,n,2}(r)\right]\end{matrix}\right\},$$

$$\overline{U}_{1\,n,3}(r)=\left\{\overline{x}_n+h\begin{bmatrix}\frac{4}{9}\left[F_2(t_n,u_1)+v_1+w_1\right]+\\ \frac{1}{18}\left[F_2(t_n+\frac{h}{2},u_2)+v_2+w_2\right]\end{bmatrix}\middle|\begin{matrix}u_1\in[\underline{U}_{1\,n,1}(r),\overline{U}_{1\,n,1}(r)],u_2\in[\underline{U}_{1\,n,2}(r),\overline{U}_{1\,n,2}(r)],\\ w_1\in[\underline{Z}_{1\,n,1}(r),\overline{Z}_{1\,n,1}(r)],w_2\in[\underline{Z}_{1\,n,2}(r),\overline{Z}_{1\,n,2}(r)]\end{matrix}\right\},$$

$$\underline{Z}_{1\,n,3}(r)=\left\{h\begin{bmatrix}\frac{4}{9}V_1(t_n,s_n,u_1)+\\ \frac{1}{18}V_1(t_n+\frac{h}{2},s_n+\frac{h}{2},u_2)\end{bmatrix}\middle|\begin{matrix}u_1\in\left[\underline{U}_{1\,n,1}(r),\overline{U}_{1\,n,1}(r)\right],\\ u_2\in\left[\underline{U}_{1\,n,2}(r),\overline{U}_{1\,n,2}(r)\right]\end{matrix}\right\},$$

$$\overline{Z}_{1\,n,3}(r)=\left\{h\begin{bmatrix}\frac{4}{9}V_2(t_n,s_n,u_1)+\\ \frac{1}{18}V_2(t_n+\frac{h}{2},s_n+\frac{h}{2},u_2)\end{bmatrix}\middle|\begin{matrix}u_1\in\left[\underline{U}_{1\,n,1}(r),\overline{U}_{1\,n,1}(r)\right],\\ u_2\in\left[\underline{U}_{1\,n,2}(r),\overline{U}_{1\,n,2}(r)\right]\end{matrix}\right\},$$

$$\underline{U}_{1\,n,4}(r)=\left\{\underline{x}_n+h\left[F_1(t_n+\frac{h}{2},u)+v_3+w\right]\middle|u\in\left[\underline{U}_{1\,n,3}(r),\overline{U}_{1\,n,3}(r)\right],w\in\left[\underline{Z}_{1\,n,3}(r),\overline{Z}_{1\,n,3}(r)\right]\right\},$$

$$\overline{U}_{1\,n,4}(r)=\left\{\overline{x}_n+h\left[F_2(t_n+\frac{h}{2},u)+v_3+w\right]\middle|u\in\left[\underline{U}_{1\,n,3}(r),\overline{U}_{1\,n,3}(r)\right],w\in\left[\underline{Z}_{1\,n,3}(r),\overline{Z}_{1\,n,3}(r)\right]\right\},$$

$$\underline{Z}_{1n,4}(r)=\left\{h\left[V_1\left(t_n+\frac{h}{2},s_n+\frac{h}{2},u\right)\right]\Big|u\in\left[\underline{U}_{1n,3}(r),\overline{U}_{1n,3}(r)\right]\right\},$$

$$\overline{Z}_{1n,4}(r)=\left\{h\left[V_2\left(t_n+\frac{h}{2},s_n+\frac{h}{2},u\right)\right]\Big|u\in\left[\underline{U}_{1n,3}(r),\overline{U}_{1n,3}(r)\right]\right\},$$

$$v_1\in\left[\underline{z}_{1n}(t_n;r),\overline{z}_{1n}(t_n;r)\right]=\left[\frac{h}{6}V_1(t_n,s_n,u),\frac{h}{6}V_2(t_n,s_n,u)\Big|u\in\left[\underline{U}_{1n,1}(r),\overline{U}_{1n,1}(r)\right]\right],$$

$$v_2\in\left[\underline{z}_{1n}\left(t_n+\frac{h}{2};r\right),\overline{z}_{1n}\left(t_n+\frac{h}{2};r\right)\right]$$
$$=\left[\frac{2h}{3}V_1\left(t_n+\frac{h}{2},s_n+\frac{h}{2},u\right),\frac{2h}{3}V_2\left(t_n+\frac{h}{2},s_n+\frac{h}{2},u\right)\Big|u\in\left[\underline{U}_{1n,2}(r),\overline{U}_{1n,2}(r)\right]\right],$$

$$v_3\in\left[\underline{z}_{1n}(t_n+\frac{h}{2};r),\overline{z}_{1n}(t_n+\frac{h}{2};r)\right]$$
$$=\left[\frac{h}{18}V_1\left(t_n+\frac{h}{2},s_n+\frac{h}{2},u\right),\frac{h}{18}V_2\left(t_n+\frac{h}{2},s_n+\frac{h}{2},u\right)\Big|u\in\left[\underline{U}_{1n,3}(r),\overline{U}_{1n,3}(r)\right]\right],$$

$$v_4\in\left[\underline{z}_{1n}(t_n+h;r),\overline{z}_{1n}(t_n+h;r)\right]$$
$$=\left[\frac{h}{6}V_1(t_n+h,s_n+h,u),\frac{h}{6}V_2(t_n+h,s_n+h,u)\Big|u\in\left[\underline{U}_{1n,4}(r),\overline{U}_{1n,4}(r)\right]\right]$$

$$\underline{y_1}_{n+1}(t;r)=\left\{\begin{array}{l}\underline{y_1}_n+\frac{h}{9}\left[F_1(t_n,u_1)+v_1+w_1\right]\\+\frac{2h}{3}\left[F_1(t_n+\frac{h}{2},u_2)+v_2+w_2\right]\\+\frac{h}{18}\left[F_1(t_n,u_3)+v_3+w_3\right]\\+\frac{h}{6}\left[F_1(t_n+h,u_4)+v_4+w_4\right]\end{array}\left|\begin{array}{l}u_1\in\left[\underline{U_1}_{n,1}(r),\overline{U}_{1n,1}(r)\right],u_2\in\left[\underline{U_1}_{n,2}(r),\overline{U}_{1n,2}(r)\right],\\u_3\in\left[\underline{U_1}_{n,3}(r),\overline{U}_{1n,3}(r)\right],u_4\in\left[\underline{U_1}_{n,4}(r),\overline{U}_{1n,4}(r)\right],\\v_1\in\left[\underline{z_1}_n(t_n;r),\overline{z}_{1n}(t_n;r)\right],\\v_2\in\left[\underline{z_1}_n\left(t_n+\frac{h}{2};r\right),\overline{z}_{1n}\left(t_n+\frac{h}{2};r\right)\right],\\v_3\in\left[\underline{z_1}_n(t_n;r),\overline{z}_{1n}(t_n;r)\right],\\v_4\in\left[\underline{z_1}_n(t_n+h;r),\overline{z}_{1n}(t_n+h;r)\right],\\w_1\in\left[\underline{Z_1}_{n,1}(r),\overline{Z}_{1n,1}(r)\right],w_2\in\left[\underline{Z_1}_{n,2}(r),\overline{Z}_{1n,2}(r)\right],\\w_3\in\left[\underline{Z_1}_{n,3}(r),\overline{Z}_{1n,3}(r)\right],w_4\in\left[\underline{Z_1}_{n,4}(r),\overline{Z}_{1n,4}(r)\right]\end{array}\right.\right\}$$

$$\overline{y}_{1n+1}(t;r)=\left\{\begin{array}{l}\overline{y}_{1n}+\frac{h}{9}\left[F_2(t_n,u_1)+v_1+w_1\right]\\+\frac{2h}{3}\left[F_2\left(t_n+\frac{h}{2},u_2\right)+v_2+w_2\right]\\+\frac{h}{18}\left[F_2(t_n,u_3)+v_3+w_3\right]\\+\frac{h}{6}\left[F_2(t_n+h,u_4)+v_4+w_4\right]\end{array}\left|\begin{array}{l}u_1\in\left[\underline{U_1}_{n,1}(r),\overline{U}_{1n,1}(r)\right],u_2\in\left[\underline{U_1}_{n,2}(r),\overline{U}_{1n,2}(r)\right],\\u_3\in\left[\underline{U_1}_{n,3}(r),\overline{U}_{1n,3}(r)\right],u_4\in\left[\underline{U_1}_{n,4}(r),\overline{U}_{1n,4}(r)\right],\\v_1\in\left[\underline{z_1}_n(t_n;r),\overline{z}_{1n}(t_n;r)\right],\\v_2\in\left[\underline{z_1}_n\left(t_n+\frac{h}{2};r\right),\overline{z}_{1n}\left(t_n+\frac{h}{2};r\right)\right],\\v_3\in\left[\underline{z_1}_n(t_n;r),\overline{z}_{1n}(t_n;r)\right],\\v_4\in\left[\underline{z_1}_n(t_n+h;r),\overline{z}_{1n}(t_n+h;r)\right],\\w_1\in\left[\underline{Z_1}_{n,1}(r),\overline{Z}_{1n,1}(r)\right],w_2\in\left[\underline{Z_1}_{n,2}(r),\overline{Z}_{1n,2}(r)\right],\\w_3\in\left[\underline{Z_1}_{n,3}(r),\overline{Z}_{1n,3}(r)\right],w_4\in\left[\underline{Z_1}_{n,4}(r),\overline{Z}_{1n,4}(r)\right]\end{array}\right.\right\}$$

for the (2) differentiability

$$\underline{U}_{2\,n,1}(r)=\left\{u\,|\,u\in\left[\underline{y}_n\left(t_n;r\right),\overline{y}_n\left(t_n;r\right)\right]\right\};\quad \overline{U}_{2\,n,1}(r)=\left\{u\,|\,u\in\left[\underline{y}_n\left(t_n;r\right),\overline{y}_n\left(t_n;r\right)\right]\right\};$$

$$\underline{Z}_{2\,n,1}(r)=0,\qquad \overline{Z}_{2\,n,1}(r)=0,$$

$$\underline{U}_{2\,n,2}(r)=\left\{\underline{y}_n+\frac{h}{2}\left[F_2(t_n,u)+v_1+w\right]\Big|\,u\in\left[\underline{U}_{2\,n,1}(r),\overline{U}_{2\,n,1}(r)\right],w\in\left[\underline{Z}_{2\,n,1}(r),\overline{Z}_{2\,n,1}(r)\right]\right\},$$

$$\overline{U}_{2\,n,2}(r)=\left\{\overline{y}_n+\frac{h}{2}\left[F_1(t_n,u)+v_1+w\right]|\,u\in\left[\underline{U}_{2\,n,1}(r),\overline{U}_{2\,n,1}(r)\right],w\in\left[\underline{Z}_{2\,n,1}(r),\overline{Z}_{2\,n,1}(r)\right]\right\},$$

$$\underline{Z}_{2\,n,2}(r)=\left\{\frac{h}{2}\left[V_2\left(t_n+\frac{h}{2},s_n+\frac{h}{2},u\right)\right]|\,u\in\left[\underline{U}_{2\,n,1}(r),\overline{U}_{2\,n,1}(r)\right],\right\},$$

$$\overline{Z}_{2\,n,2}(r)=\left\{\frac{h}{2}\left[V_1\left(t_n+\frac{h}{2},s_n+\frac{h}{2},u\right)\right]|\,u\in\left[\underline{U}_{2\,n,1}(r),\overline{U}_{2\,n,1}(r)\right]\right\},$$

$$\underline{U}_{2\,n,3}(r)=\left\{\underline{x}_n+h\begin{bmatrix}\frac{4}{9}\left[F_2(t_n,u_1)+v_1+w_1\right]\\ +\frac{1}{18}\left[F_2(t_n+\frac{h}{2},u_2)+v_2+w_2\right]\end{bmatrix}\middle|\begin{matrix}u_1\in\left[\underline{U}_{2\,n,1}(r),\overline{U}_{2\,n,1}(r)\right],u_2\in\left[\underline{U}_{2\,n,2}(r),\overline{U}_{2\,n,2}(r)\right],\\ w_1\in\left[\underline{Z}_{2\,n,1}(r),\overline{Z}_{2\,n,1}(r)\right],w_2\in\left[\underline{Z}_{2\,n,2}(r),\overline{Z}_{2\,n,2}(r)\right]\end{matrix}\right\},$$

$$\overline{U}_{2\,n,3}(r)=\left\{\overline{y}_n+h\begin{bmatrix}\frac{4}{9}\left[F_1(t_n,u_1)+v_1+w_1\right]\\ +\frac{1}{18}\left[F_1(t_n+\frac{h}{2},u_2)+v_2+w_2\right]\end{bmatrix}\middle|\begin{matrix}u_1\in\left[\underline{U}_{2\,n,1}(r),\overline{U}_{1\,n,1}(r)\right],u_2\in\left[\underline{U}_{2\,n,2}(r),\overline{U}_{1\,n,2}(r)\right],\\ w_1\in\left[\underline{Z}_{2\,n,1}(r),\overline{Z}_{1\,n,1}(r)\right],w_2\in\left[\underline{Z}_{2\,n,2}(r),\overline{Z}_{1\,n,2}(r)\right]\end{matrix}\right\},$$

$$\underline{Z}_{2\,n,3}(r)=\left\{h\begin{bmatrix}\frac{4}{9}V_2(t_n,s_n,u_1)\\ +\frac{1}{18}V_2\left(t_n+\frac{h}{2},s_n+\frac{h}{2},u_2\right)\end{bmatrix}\middle|\begin{matrix}u_1\in\left[\underline{U}_{2\,n,1}(r),\overline{U}_{2\,n,1}(r)\right],\\ u_2\in\left[\underline{U}_{2\,n,2}(r),\overline{U}_{2\,n,2}(r)\right]\end{matrix}\right\},$$

$$\overline{Z}_{2\,n,3}(r)=\left\{h\begin{bmatrix}\frac{4}{9}V_1(t_n,s_n,u_1)\\ +\frac{1}{18}V_1\left(t_n+\frac{h}{2},s_n+\frac{h}{2},u_2\right)\end{bmatrix}\middle|\begin{matrix}u_1\in\left[\underline{U}_{2\,n,1}(r),\overline{U}_{2\,n,1}(r)\right],\\ u_2\in\left[\underline{U}_{2\,n,2}(r),\overline{U}_{2\,n,2}(r)\right]\end{matrix}\right\},$$

$$\underline{U_2}_{n,4}(r)=\left\{\underline{y}_n+h\left[F_2\left(t_n+\frac{h}{2},u\right)+v_3+w\right]\middle|\begin{matrix}u\in\left[\underline{U_2}_{n,3}(r),\overline{U_2}_{n,3}(r)\right],\\ w\in\left[\underline{Z_2}_{n,3}(r),\overline{Z_2}_{n,3}(r)\right]\end{matrix}\right\},$$

$$\overline{U_2}_{n,4}(r)=\left\{\overline{y}_n+h\left[F_1\left(t_n+\frac{h}{2},u\right)+v_3+w\right]\middle|\begin{matrix}u\in\left[\underline{U_2}_{n,3}(r),\overline{U_2}_{n,3}(r)\right],\\ w\in\left[\underline{Z_2}_{n,3}(r),\overline{Z_2}_{n,3}(r)\right]\end{matrix}\right\},$$

$$\underline{Z_2}_{n,4}(r)=\left\{h\left[V_2(t_n+\frac{h}{2},s_n+\frac{h}{2},u)\right]\middle|u\in\left[\underline{U_2}_{n,3}(r),\overline{U_2}_{n,3}(r)\right]\right\},$$

$$\overline{Z_2}_{n,4}(r)=\left\{h\left[V_2(t_n+\frac{h}{2},s_n+\frac{h}{2},u)\right]\middle|u\in\left[\underline{U_2}_{n,3}(r),\overline{U_2}_{n,3}(r)\right]\right\},$$

$$v_1\in\left[\underline{z_2}_n(t_n;r),\overline{z_2}_n(t_n;r)\right]=\left[\frac{h}{6}V_2(t_n,s_n,u),\frac{h}{6}V_1(t_n,s_n,u)\middle|u\in\left[\underline{U_2}_{n,1}(r),\overline{U_2}_{n,1}(r)\right]\right],$$

$$v_2\in\left[\underline{z_2}_n\left(t_n+\frac{h}{2};r\right),\overline{z_2}_n\left(t_n+\frac{h}{2};r\right)\right]$$

$$=\left[\frac{2h}{3}V_2\left(t_n+\frac{h}{2},s_n+\frac{h}{2},u\right),\frac{2h}{3}V_1\left(t_n+\frac{h}{2},s_n+\frac{h}{2},u\right)\middle|u\in\left[\underline{U_2}_{n,2}(r),\overline{U_2}_{n,2}(r)\right]\right],$$

$$v_3\in\left[\underline{z_2}_n\left(t_n+\frac{h}{2};r\right),\overline{z_2}_n\left(t_n+\frac{h}{2};r\right)\right]$$

$$=\left[\frac{h}{18}V_2\left(t_n+\frac{h}{2},s_n+\frac{h}{2},u\right),\frac{h}{18}V_1\left(t_n+\frac{h}{2},s_n+\frac{h}{2},u\right)\middle|u\in\left[\underline{U_2}_{n,3}(r),\overline{U_2}_{n,3}(r)\right]\right],$$

$$v_4\in\left[\underline{z_2}_n(t_n+h;r),\overline{z_2}_n(t_n+h;r)\right]$$

$$=\left[\frac{h}{6}V_2(t_n+h,s_n+h,u),\frac{h}{6}V_1(t_n+h,s_n+h,u)\,|\,u\in[\underline{U_2}_{n,4}(r),\overline{U_2}_{n,4}(r)]\right].$$

$$\underline{y}_{2n+1}(t;r)=\left\{\begin{array}{l}\underline{y}_{2n}+\dfrac{h}{9}\left[F_2(t_n,u_1)+v_1+w_1\right]\\+\dfrac{2h}{3}\left[F_2\left(t_n+\dfrac{h}{2},u_2\right)+v_2+w_2\right]\\+\dfrac{h}{18}\left[F_2(t_n,u_3)+v_3+w_3\right]\\+\dfrac{h}{6}\left[F_2(t_n+h,u_4)+v_4+w_4\right]\end{array}\middle|\begin{array}{l}u_1\in\left[\underline{U}_{2n,1}(r),\overline{U}_{2n,1}(r)\right],u_2\in\left[\underline{U}_{2n,2}(r),\overline{U}_{2n,2}(r)\right],\\u_3\in\left[\underline{U}_{2n,3}(r),\overline{U}_{2n,3}(r)\right],u_4\in\left[\underline{U}_{2n,4}(r),\overline{U}_{2n,4}(r)\right],\\v_1\in\left[\underline{z}_{2n}(t_n;r),\overline{z}_{2n}(t_n;r)\right],\\v_2\in\left[\underline{z}_{2n}\left(t_n+\dfrac{h}{2};r\right),\overline{z}_{2n}\left(t_n+\dfrac{h}{2};r\right)\right],\\v_3\in\left[\underline{z}_{2n}(t_n;r),\overline{z}_{2n}(t_n;r)\right],\\v_4\in\left[\underline{z}_{2n}(t_n+h;r),\overline{z}_{2n}(t_n+h;r)\right],\\w_1\in\left[\underline{Z}_{2n,1}(r),\overline{Z}_{2n,1}(r)\right],w_2\in\left[\underline{Z}_{2n,2}(r),\overline{Z}_{2n,2}(r)\right],\\w_3\in\left[\underline{Z}_{2n,3}(r),\overline{Z}_{2n,3}(r)\right],w_4\in\left[\underline{Z}_{2n,4}(r),\overline{Z}_{2n,4}(r)\right],\end{array}\right\}$$

$$\overline{y}_{2n+1}(t;r)=\left\{\begin{array}{l}\overline{y}_{2n}+\dfrac{h}{9}\left[F_1(t_n,u_1)+v_1+w_1\right]\\+\dfrac{2h}{3}\left[F_1\left(t_n+\dfrac{h}{2},u_2\right)+v_2+w_2\right]\\+\dfrac{h}{18}\left[F_1(t_n,u_3)+v_3+w_3\right]\\+\dfrac{h}{6}\left[F_1(t_n+h,u_4)+v_4+w_4\right]\end{array}\middle|\begin{array}{l}u_1\in\left[\underline{U}_{2n,1}(r),\overline{U}_{2n,1}(r)\right],u_2\in\left[\underline{U}_{2n,2}(r),\overline{U}_{2n,2}(r)\right],\\u_3\in\left[\underline{U}_{2n,3}(r),\overline{U}_{2n,3}(r)\right],u_4\in\left[\underline{U}_{2n,4}(r),\overline{U}_{2n,4}(r)\right],\\v_1\in\left[\underline{z}_{2n}(t_n;r),\overline{z}_{2n}(t_n;r)\right],\\v_2\in\left[\underline{z}_{2n}\left(t_n+\dfrac{h}{2};r\right),\overline{z}_{2n}\left(t_n+\dfrac{h}{2};r\right)\right],\\v_3\in\left[\underline{z}_{2n}(t_n;r),\overline{z}_{2n}(t_n;r)\right],\\v_4\in\left[\underline{z}_{2n}(t_n+h;r),\overline{z}_{2n}(t_n+h;r)\right],\\w_1\in\left[\underline{Z}_{2n,1}(r),\overline{Z}_{2n,1}(r)\right],w_2\in\left[\underline{Z}_{2n,2}(r),\overline{Z}_{2n,2}(r)\right],\\w_3\in\left[\underline{Z}_{2n,3}(r),\overline{Z}_{2n,3}(r)\right],w_4\in\left[\underline{Z}_{2n,4}(r),\overline{Z}_{2n,4}(r)\right].\end{array}\right\}$$

5. NUMERICAL EXAMPLES

This section illustrates the Runge-Kutta method of fourth order under generalized differentiability which was proposed in previous section to solve FVIDE.

EXAMPLE 5.1

Consider the linear FVIDE

$$y'(t;r)=\int_0^t y(s;r)ds, \qquad t\in[0,1],$$
$$y(0;r)=(4+2r,10-4r), \quad r\in[0,1].$$

The exact solution of the problem (8) are:

$$y_1(t;r)=\left[(4+2r)((e^{-t}+e^{t})/2),(10-4r)((e^{-t}+e^{t})/2)\right],$$

is the (1)-differentiable solution and

$$y_2(t;r)=\left[(4+2r)(-(e^{-t}+e^{t})/2),(10-4r)(-(e^{-t}+e^{t})/2)\right],$$

is the (2)-differentiable solution.

To find the approximate solutions, we divide *I* into *N=10* equally spaced subintervals and apply Runge-Kutta method of order four. The comparison of solutions at*t=0.1.* is shown in the following Figures and Tables.

Table 1

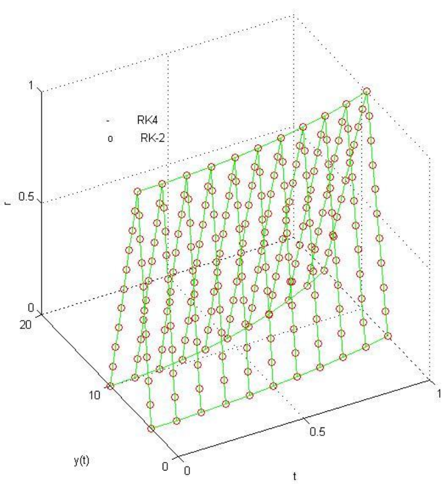
Approximate value for (1)-differentiable solutions at t=0

r	FVIDE RK2		FVIDE RK4		Exact	
	$\underline{y_1}(t;r)$	$\overline{y_1}(t;r)$	$\underline{y_1}(t;r)$	$\overline{y_1}(t;r)$	$\underline{y_1}(t;r)$	$\overline{y_1}(t;r)$
0.0	4.0000	1.0000	4.0000	1.0000	4.0000	1.0000
0.1	4.2000	9.6000	4.2000	9.6000	4.2000	9.6000
0.2	4.4000	9.2000	4.4000	9.2000	4.4000	9.2000
0.3	4.6000	8.8000	4.6000	8.8000	4.6000	8.8000
0.4	4.8000	8.4000	4.8000	8.4000	4.8000	8.4000
0.5	5.0000	8.0000	5.0000	8.0000	5.0000	8.0000
0.6	5.2000	7.6000	5.2000	7.6000	5.2000	7.6000
0.7	5.4000	7.2000	5.4000	7.2000	5.4000	7.2000
0.8	5.6000	6.8000	5.6000	6.8000	5.6000	6.8000
0.9	5.8000	6.4000	5.8000	6.4000	5.8000	6.4000
1.0	6.0000	6.0000	6.0000	6.0000	6.0000	6.0000

Table 2

Error in (1)-differentiable solutions at t=0

r	Error RK2		Error RK4	
	$\underline{y_1}(t;r)$	$\overline{y_1}(t;r)$	$\underline{y_1}(t;r)$	$\overline{y_1}(t;r)$
0.0	6.0000360235790100E-10	1.5000107822515900E-09	3.9506442561787500E-10	9.8765973177705800E-10
0.1	6.3000449301853200E-10	1.4400090009303300E-09	4.1481751367200600E-10	9.4815355566879600E-10
0.2	6.6000449550074300E-10	1.3800089959659100E-09	4.3457060172613600E-10	9.0864737956053400E-10
0.3	6.9000449798295500E-10	1.3200089910014900E-09	4.5432368978026700E-10	8.6914120345227300E-10
0.4	7.2000450046516600E-10	1.2600089860370600E-09	4.7407677783439800E-10	8.2963502734401100E-10
0.5	7.5000539112579700E-10	1.2000072047158000E-09	4.9382986588852900E-10	7.9012885123575000E-10
0.6	7.8000539360800800E-10	1.1400071997513800E-09	5.1358295394266000E-10	7.5062178694906800E-10
0.7	8.1000539609021900E-10	1.0800071947869600E-09	5.3333604199679000E-10	7.1111561084080700E-10
0.8	8.4000539857243000E-10	1.0200071898225400E-09	5.5309001822934100E-10	6.7160854655412600E-10
0.9	8.7000540105464100E-10	9.6000629667969400E-10	5.7284310628347200E-10	6.3210237044586400E-10
1.0	9.0000629171527200E-10	9.0000629171527200E-10	5.9259619433760200E-10	5.9259619433760200E-10



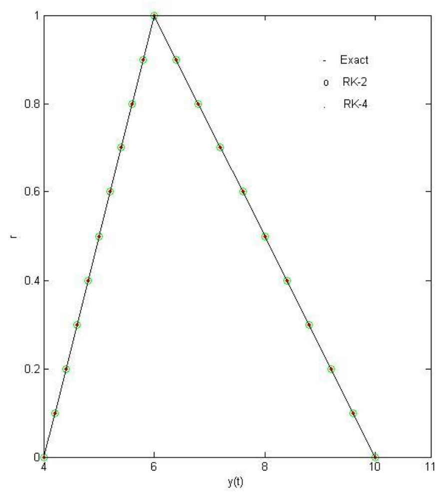


Figure 1: The exact and approximate (1)-differentiable solutions of FVIDE by Runge-Kutta method of order 4.

Table 3

Approximate value for (2)-differentiable solutions at t=0

r	FVIDE		Exact		Exact	
	$\underline{y_2}(t;r)$	$\overline{y_2}(t;r)$	$\underline{y_2}(t;r)$	$\overline{y_2}(t;r)$	$\underline{y_2}(t;r)$	$\overline{y_2}(t;r)$
0.0	-4.0000	-1.0000	-4.0000	-1.0000	-4.0000	-1.0000
0.1	-4.2000	-9.6000	-4.2000	-9.6000	4.2000	-9.6000
0.2	-4.4000	-9.2000	-4.4000	-9.2000	4.4000	-9.2000
0.3	-4.6000	-8.8000	-4.6000	-8.8000	-4.6000	-8.8000
s0.4	-4.8000	-8.4000	-4.8000	-8.4000	-4.8000	-8.4000
0.5	-5.0000	-8.0000	-5.0000	-8.0000	-5.0000	-8.0000
0.6	-5.2000	-7.6000	-5.2000	-7.6000	-5.2000	-7.6000
0.7	-5.4000	-7.2000	-5.4000	-7.2000	-5.4000	-7.2000
0.8	-5.6000	-6.8000	-5.6000	-6.8000	-5.6000	-6.8000
0.9	-5.8000	-6.4000	-5.8000	-6.4000	-5.8000	-6.4000
1.0	-6.0000	-6.0000	-6.0000	-6.0000	-6.0000	-6.0000

Table 4

Error in (2)-differentiable solutions at t=0

R	Error RK2		Error RK4	
	$\underline{y_2}(t;r)$	$\overline{y_2}(t;r)$	$\underline{y_2}(t;r)$	$\overline{y_2}(t;r)$
0.0	9.0000717989369200E-10	1.2000072047158000E-09	9.8765973177705800E-10	3.9506353743945500E-10
0.1	9.0000717989369200E-10	1.1700063140551700E-09	9.4815355566879600E-10	4.1481662549358600E-10
0.2	9.0000717989369200E-10	1.1400071997513800E-09	9.0864737956053400E-10	4.3456971354771700E-10
0.3	9.0000629171527200E-10	1.1100063090907500E-09	8.6914120345227300E-10	4.5432280160184700E-10
0.4	9.0000629171527200E-10	1.0800071947869600E-09	8.2963413916559200E-10	4.7407766601281800E-10
0.5	9.0000629171527200E-10	1.0500063041263300E-09	7.9012796305733000E-10	4.9383075406694800E-10
0.6	9.0000629171527200E-10	1.0200063016441200E-09	7.5062178694906800E-10	5.1358295394266000E-10
0.7	9.0000629171527200E-10	9.9000629916190500E-10	7.1111561084080700E-10	5.3333604199679000E-10
0.8	9.0000629171527200E-10	9.6000629667969400E-10	6.7160854655412600E-10	5.5309001822934100E-10
0.9	9.0000629171527200E-10	9.3000629419748300E-10	6.3210237044586400E-10	5.7284310628347200E-10

1.0	9.0000629171527200E-10	9.0000629171527200E-10	5.9259619433760200E-10	5.9259619433760200E-10

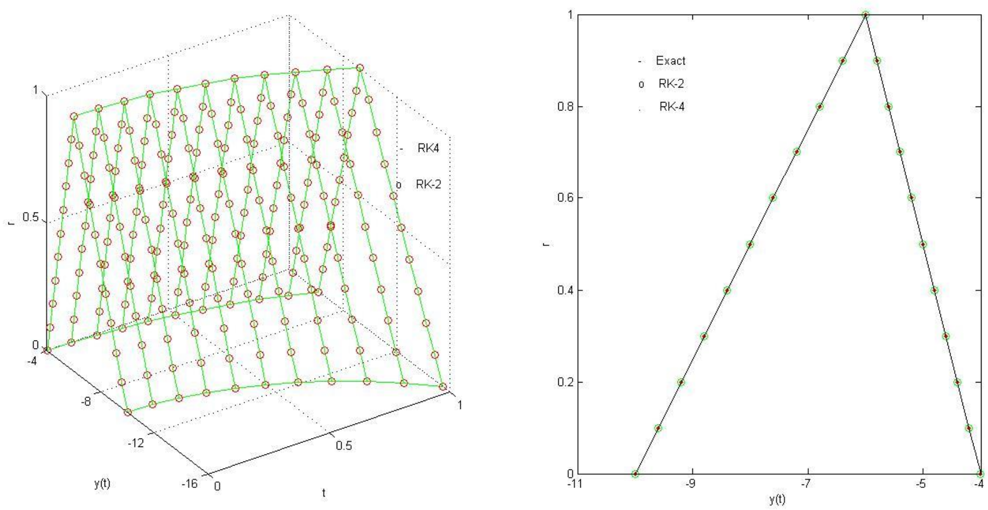


Figure 2: The exact and approximate (2)-differentiable solutions of FVIDE by Runge-Kutta method of order 4.

EXAMPLE 5.2

Consider the linear FVIDE

$$y'(t;r) = \int_0^t y(s;r)ds, \qquad t \in \left[0, \frac{\sqrt{2}}{2}\right],$$

$$y(0;r) = (1+0.5r, 2-0.5r), \quad r \in [0,1]. \quad (9)$$

The exact solution of the problem (9) are:

$$y_1(t;r) = \left[\cos(t)(1+0.5r) - \frac{3}{2}\cos(t) + \frac{3}{4}(e^{-t}+e^t), \cos(t)(2-0.5r) - \frac{3}{2}\cos(t) + \frac{3}{4}(e^{-t}+e^t)\right],$$

is the (1)-differentiable solution andss

$$y_2(t;r) = \left[-\cos(t)(1+0.5r) - \frac{3}{2}\cos(t) + \frac{3}{4}(e^{-t}+e^t), -\cos(t)(2-0.5r) - \frac{3}{2}\cos(t) + \frac{3}{4}(e^{-t}+e^t)\right],$$

is the (2)-differentiable solution.

To find the approximate solutions, we divide *I* into *N=10* equally spaced subintervals and apply Runge-Kutta method of order four. The comparison of solutions at *t=0.1* is shown in the following Figures and Tables.

Table 5

Approximate value for (1)-differentiable solutions at t=0

r	FVIDE		Exact		Exact	
	$\underline{y_1}(t;r)$	$\overline{y_1}(t;r)$	$\underline{y_1}(t;r)$	$\overline{y_1}(t;r)$	$\underline{y_1}(t;r)$	$\overline{y_1}(t;r)$
0.0	1.0000	2.0000	1.0000	2.0000	1.0000	2.0000
0.1	1.0500	1.9500	1.0500	1.9500	1.0500	1.9500
0.2	1.1000	1.9000	1.1000	1.9000	1.1000	1.9000
0.3	1.1500	1.8500	1.1500	1.8500	1.1500	1.8500
0.4	1.2000	1.8000	1.2000	1.8000	1.2000	1.8000
0.5	1.2500	1.7500	1.2500	1.7500	1.2500	1.7500
0.6	1.3000	1.7000	1.3000	1.7000	1.3000	1.7000
0.7	1.3500	1.6500	1.3500	1.6500	1.3500	1.6500
0.8	1.4000	1.6000	1.4000	1.6000	1.4000	1.6000
0.9	1.4500	1.5500	1.4500	1.5500	1.4500	1.5500
1.0	1.5000	1.5000	1.5000	1.5000	1.5000	1.5000

Table 6

Error in (1)-differentiable solutions at t=0

r	Error RK2		Error RK4	
	$\underline{y_1}(t;r)$	$\overline{y_1}(t;r)$	$\underline{y_1}(t;r)$	$\overline{y_1}(t;r)$
0.0	1.5000090058947500E-10	3.0000180117895100E-10	9.8766106404468700E-11	1.9753221280893700E-10
0.1	1.5750112325463300E-10	2.9250202260300300E-10	1.0370437841800100E-10	1.9259371875080000E-10
0.2	1.6500112387518600E-10	2.8500179993784500E-10	1.0864265043153400E-10	1.8765544673726700E-10
0.3	1.7250112449573900E-10	2.7750179931729200E-10	1.1358092244506700E-10	1.8271717472373400E-10
0.4	1.8000112511629100E-10	2.7000179869674000E-10	1.1851919445859900E-10	1.7777890271020200E-10
0.5	1.8750134778144900E-10	2.6250179807618700E-10	1.2345746647213200E-10	1.7284063069666900E-10
0.6	1.9500134840200200E-10	2.5500179745563400E-10	1.2839573848566500E-10	1.6790213663853100E-10
0.7	2.0250134902255500E-10	2.4750157479047600E-10	1.3333401049919800E-10	1.6296386462499900E-10
0.8	2.1000134964310700E-10	2.4000157416992400E-10	1.3827250455733500E-10	1.5802559261146600E-10
0.9	2.1750135026366000E-10	2.3250157354937100E-10	1.4321077657086800E-10	1.5308732059793300E-10
1.0	2.2500157292881800E-10	2.2500157292881800E-10	1.4814904858440100E-10	1.4814904858440100E-10

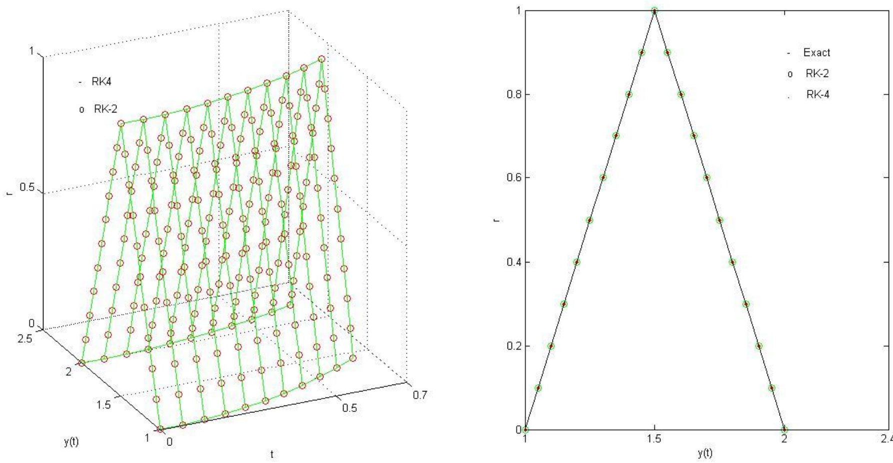


Figure 3: The exact and approximate (1)-differentiable solutions of FVIDE by Runge-Kutta method of order 4.

Table 7

Approximate value for (2)-differentiable solutions at t=0

r	FVIDE		Exact		Exact	
	$\underline{y_2}(t;r)$	$\overline{y_2}(t;r)$	$\underline{y_2}(t;r)$	$\overline{y_2}(t;r)$	$\underline{y_2}(t;r)$	$\overline{y_2}(t;r)$
0.0	-1.0000	-2.0000	-1.0000	-2.0000	-1.0000	-2.0000
0.1	-1.0500	-1.9500	-1.0500	-1.9500	-1.0500	-1.9500
0.2	-1.1000	-1.9000	-1.1000	-1.9000	-1.1000	-1.9000
0.3	-1.1500	-1.8500	-1.1500	-1.8500	-1.1500	-1.8500
0.4	-1.2000	-1.8000	-1.2000	-1.8000	-1.2000	-1.8000
0.5	-1.2500	-1.7500	-1.2500	-1.7500	-1.2500	-1.7500
0.6	-1.3000	-1.7000	-1.3000	-1.7000	-1.3000	-1.7000
0.7	-1.3500	-1.6500	-1.3500	-1.6500	-1.3500	-1.6500
0.8	-1.4000	-1.6000	-1.4000	-1.6000	-1.4000	-1.6000
0.9	-1.4500	-1.5500	-1.4500	-1.5500	-1.4500	-1.5500
1.0	-1.5000	-1.5000	-1.5000	-1.5000	-1.5000	-1.5000

Table 8

Error in (2)-differentiable solutions at t=0

r	Error RK2		Error RK4	
	$\underline{y_2}(t;r)$	$\overline{y_2}(t;r)$	$\underline{y_2}(t;r)$	$\overline{y_2}(t;r)$
0.0	2.5000157499732700E-10	2.0000134881570400E-10	1.9753199076433200E-10	9.8765884359863800E-11
0.1	2.4750157479047600E-10	2.0250157106716000E-10	1.9259371875080000E-10	1.0370437841800100E-10
0.2	2.4500157458362500E-10	2.0500157127401100E-10	1.8765544673726700E-10	1.0864265043153400E-10
0.3	2.4250157437677400E-10	2.0750157148086100E-10	1.8271717472373400E-10	1.1358092244506700E-10
0.4	2.4000157416992400E-10	2.1000157168771200E-10	1.7777890271020200E-10	1.1851919445859900E-10
0.5	2.3750157396307300E-10	2.1250157189456300E-10	1.7284063069666900E-10	1.2345746647213200E-10
0.6	2.3500157375622200E-10	2.1500157210141400E-10	1.6790213663853100E-10	1.2839573848566500E-10
0.7	2.3250157354937100E-10	2.1750157230826500E-10	1.6296386462499900E-10	1.3333401049919800E-10
0.8	2.3000157334252000E-10	2.2000157251511600E-10	1.5802559261146600E-10	1.3827250455733500E-10
0.9	2.2750157313566900E-10	2.2250157272196700E-10	1.5308732059793300E-10	1.4321077657086800E-10
1.0	2.2500157292881800E-10	2.2500157292881800E-10	1.4814904858440100E-10	1.4814904858440100E-10

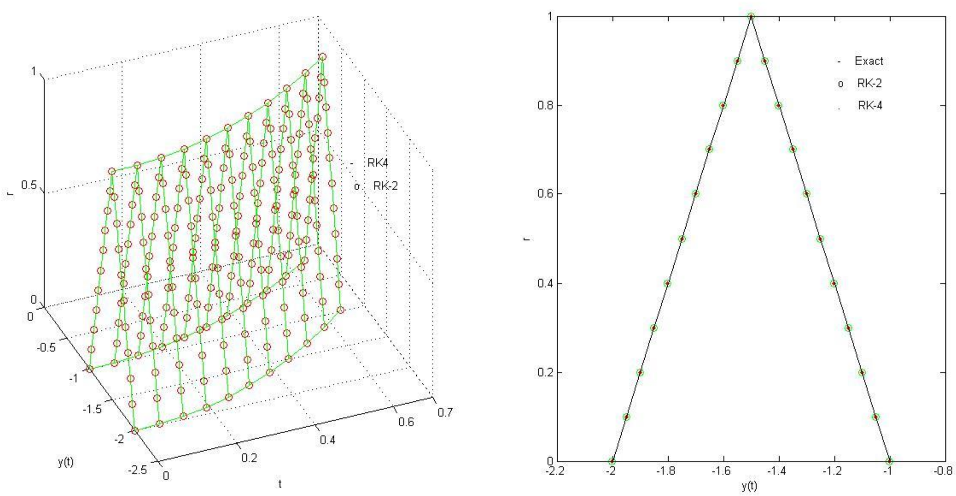


Figure 4: The exact and approximate (2)-differentiable solutions of FVIDE by Runge-Kutta method of order 4.

6. CONCLUSION

In this paper we have used fourth order Runge-Kutta method for solving FVIDE under generalized differentiability. From the examples provided in section 5, we see that the approximate solutions by Runge-Kutta method of order four coincide with the exact solutions. This work which presents applicable model for improved approximation which suits the FVIDE theoretical findings with the real time applications. Higher order Runge-Kutta methods will be considered in future work.

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